Equilibrium Similarity and Turbulence at Finite Reynolds Number

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The classical theory tells us (and most modern texts as well) all jets are alike asymptotically – only the initial momentum matters.

large scales: energy containing eddies

small scales: energy dissipation (1mm or less)

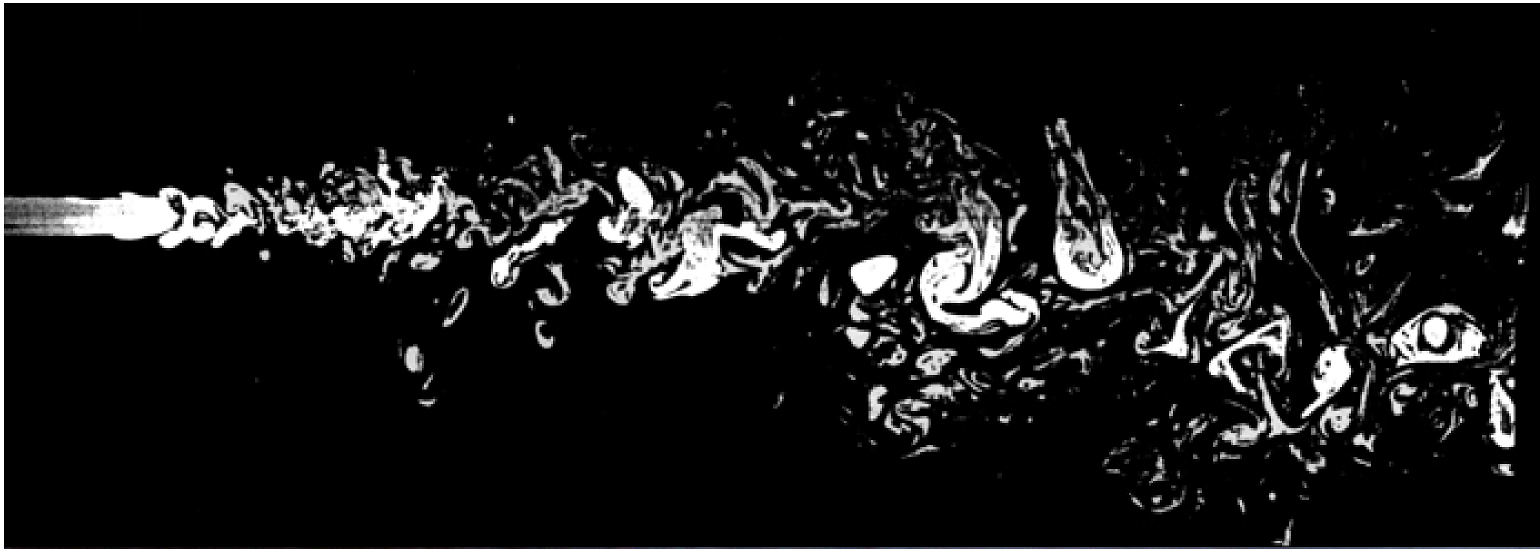
High Re ⟶ large scale separation

Even the Reynolds number has been believed to be nearly irrelevant.

But jets really don’t all look alike.

And Reynolds number certainly plays a role – especially if it is too small.

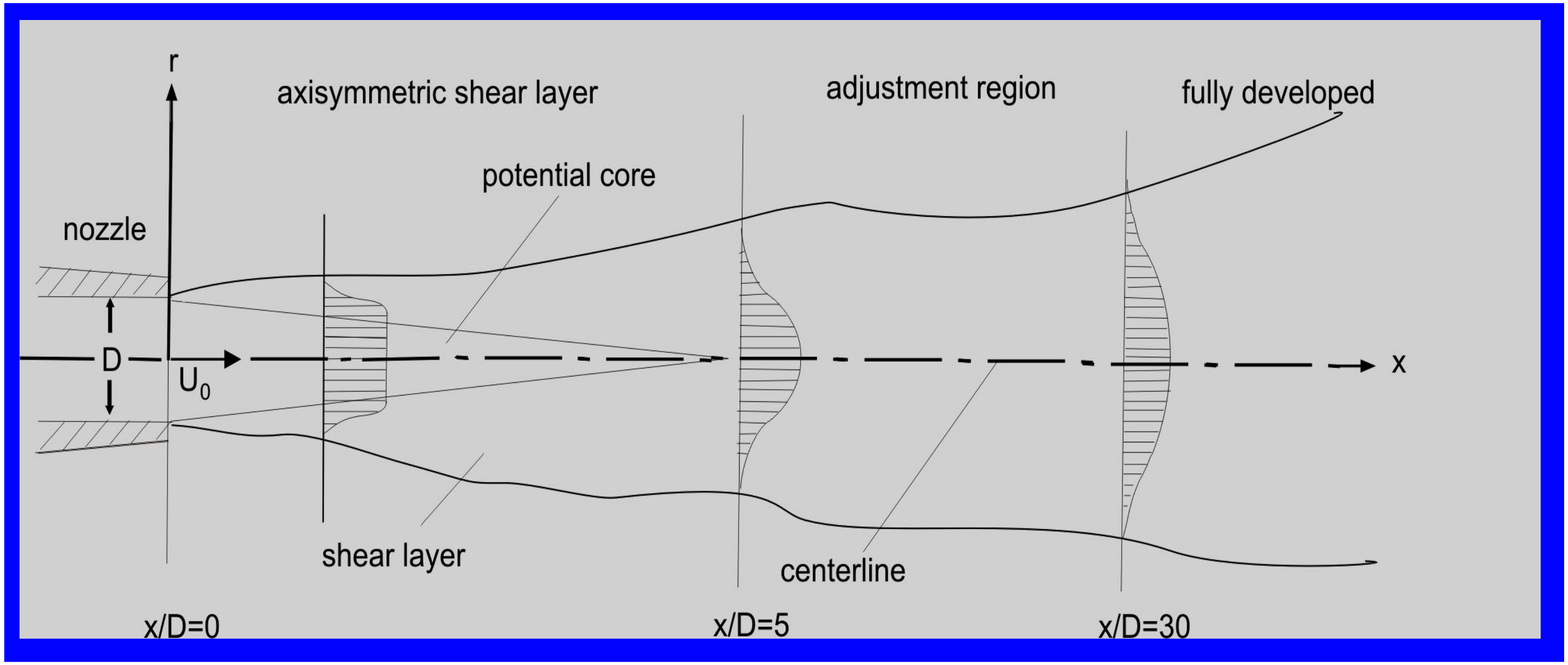
Low Reynolds # 2 300



High Reynolds # 65 000

Equilibrium similarity provides a way to understand this...

- Consider the fully-developed axisymmetric jet



$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial r}=\frac{1}{r}\frac{\partial}{\partial r}r<-uv>, \qquad \frac{\partial U}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}rV=0$$

SINGLE POINT EQUILIBRIUM SIMILARITY

- Change variables to:

$$U = U_s(x) f(\eta), \quad -<uv>= R_s(x) g(\eta)$$

etc., where: $\eta = r/\delta(x)$

- Momentum equation for jet transforms

to:

$$\left[U_s \frac{dU_s}{dx}\right] f^2 - \left\{2\left[\frac{U_s^2}{\delta}\frac{d\delta}{dx}\right] + \left[U_s \frac{dU_s}{dx}\right]\right\} f' \int_0^\eta f(\bar{\eta})\bar{\eta}\, d\bar{\eta}$$

All explicit x-depen-dence in square brackets.

$$= \left[\frac{R_s(x)}{\delta}\right]\frac{(\eta g)'}{\eta}$$

EQUILIBRIUM SIMILARITY HYPOTHESIS

- All of the terms in square brackets of the equation must evolve with downstream distance in exactly the same way (unless they are identically zero).

- There are no further assumptions.

- If no solution consistent with boundary conditions, none will be found.

$$\left[U_s \frac{dU_s}{dx}\right] f^2 - \left\{ 2\left[\frac{U_s^2}{\delta}\frac{d\delta}{dx}\right] + \left[U_s \frac{dU_s}{dx}\right]\right\} f' \int_0^{\eta} f(\overline{\eta})\overline{\eta}\, d\overline{\eta} = \left[\frac{R_s(x)}{\delta}\right]\frac{(\eta g)'}{\eta}$$

- If we divide by second term, momentum equation for jet reduces to:

$$-f^2 - f' \int_0^{\eta} f(\overline{\eta})\overline{\eta}\, d\overline{\eta} = \left[\frac{R_s(x)}{U_s^2(x) d\delta(x)/dx}\right]\frac{(\eta g)'}{\eta}$$

- The single bracketed term remaining can at most depend on the upstream conditions.

$$-f^2 - f' \int_0^{\eta} f(\overline{\eta})\overline{\eta}\, d\overline{\eta} = \left[\frac{R_s(x)}{U_s^2(x) d\delta(x)/dx} \right] \frac{(\eta g)'}{\eta}$$

- We can redefine the Reynolds stress profile function g to absorb the constant; i.e.,

$$-f^2 - f' \int_0^{\eta} f(\overline{\eta})\overline{\eta}\, d\overline{\eta} = (\eta \tilde{g})' / \eta$$

- **Clearly the solutions are dependent only on η, since there are no parameters left.**

Mean momentum equation:

$$-f^2 - f'\int_0^{\eta} f(\overline{\eta})\overline{\eta}\,d\overline{\eta} = (\eta\tilde{g})'/\eta$$

- Generally not possible to do this for Reynolds stress equations.
- So, profiles of turbulence quantities will be different for different upstream conditions.

- Thus mean velocity and *properly scaled* Reynolds shear stress ***profiles*** should be universal.

- BUT the growth rate $d\delta / dx$ and centerline velocity decay U_c can reflect ***source*** conditions, i.e.,

Half-width: $\delta_{1/2} = b\,(x - x_o)$;

Centerline velocity: $U_c = a\,\left(x - x_o\right)^{-1}$

- Constants *a* and *b* depend on source conditions (c.f., Grinstein (2001), Boersma et al. 1998, Slessor et al. (1998), Nobis et al. 2001, Cater & Soria 2002, Westerweel et al. 2002).

- Even in late 80's, it was clear the two-point equations must play a major role.
- Dissipation scaling determined (or reflected) the spreading rate and its dependence on source Reynolds number:

$$\varepsilon \propto U_c^3 / \delta_{1/2} \qquad vs \qquad \varepsilon \propto \nu U_c^2 / \delta_{1/2}^2$$

- Differences obscured by fact that turbulent jet evolves at constant Re, no matter the source Reynolds number i.e., $\mathrm{Re} = U_c \delta_{1/2} / \nu = const$
- Consequence of momentum conservation, $M=M_o$.

This opens interesting new opportunities:

- Complete similarity analysis of the *two-point* Reynolds stress equations possible since axisymmetric jet evolves at constant local Reynolds number, $U_c\delta/\nu$ (Ewing 1995).

- Streamwise growth is removed by stretching the coordinate system logarithmically.

- Similarity variables are:

$$\xi = \ln (x - x_o)/D, \quad \eta = r/\delta(x), \ \theta$$

- The two-point equations are quite complicated, and even resulting constraints are not trivial...

$$\left[U_s(x)\frac{\partial Q^{i,j}}{\partial x_1}\right] \propto \left[\frac{Q^{i,j}U_s(x)}{x_1}\right] \propto \left[\frac{\partial \Pi_1^j}{\partial x_1}\right]\delta_{i1} \propto \left[\frac{\Pi_1^j}{x_1}\right]\delta_{i1} \propto \left[\frac{\Pi_1^j}{\delta(x_1)}\right](\delta_{i2}+\delta_{i3})$$

$$\propto \left[\frac{\partial T_1^{1i,j}}{\partial x_1}\right] \propto \left[\frac{T_1^{1i,j}}{x_1}\right] \propto \left[\frac{T_1^{2i,j}}{\delta(x_1)}\right] \propto \left[\frac{T_1^{3i,j}}{\delta(x_1)}\right]$$

$$\propto \left[\frac{Q^{1,j}U_s}{x_1}\right]\delta_{i1} \propto \left[\frac{Q^{2,j}U_s}{\delta(x_1)}\right]\delta_{i1} \propto \left[\left[\frac{Q^{1,j}U_s}{x_1}\right]\frac{d\delta}{dx_1}\right]\delta_{i2} \propto \left[\left[\frac{Q^{2,j}U_s}{x_1}\right]\right]\delta_{i2}$$

$$\propto \nu\left[\frac{\partial^2 Q^{i,j}}{\partial x_1{}^2}\right] \propto \nu\left[\frac{2Q^{i,j}}{x_1{}^2}\right] \propto \nu\left[\frac{2}{x_1}\frac{\partial Q^{i,j}}{\partial x_1}\right] \propto \nu\left[\frac{Q^{i,j}}{\delta^2}\right]$$

$$\propto \nu\left[\frac{Q^{2,j}}{\delta^2}\right]\delta_{i3} \propto \nu\left[\frac{Q^{3,j}}{\delta^2}\right]\delta_{i2} \propto \nu\left[\frac{Q^{3,j}}{\delta^2}\right]\delta_{i3} \propto \nu\left[\frac{2Q^{2,j}}{\delta^2}\right]\delta_{i3}$$

PLUS an additional set for the other point, x_1' .

Two-point velocity correlation tensor become

$$< u_i(x,r,\theta,t)u_j(x',r',\theta',t) >$$

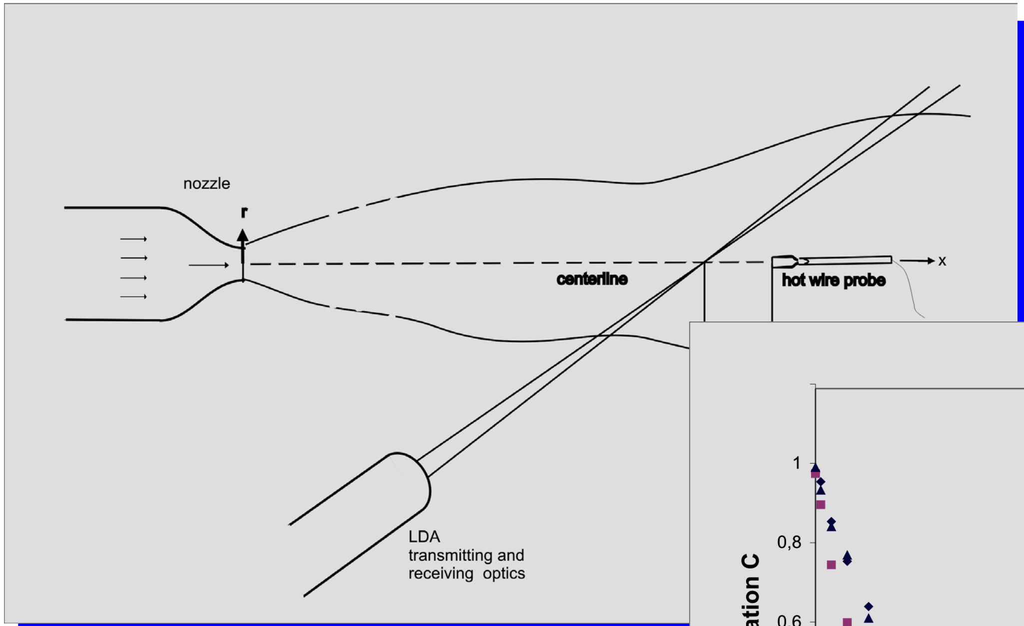
$$= U_c(x)U_c(x')\left[\frac{d\delta_{1/2}}{dx_1}\frac{d\delta'_{1/2}}{dx'_1}\right]^{1-\delta_{ij}} Q_{i,j}$$

where $Q_{i,j} = Q_{i,j}(\xi'-\xi,\ \eta, \eta',\ \theta'-\theta),$

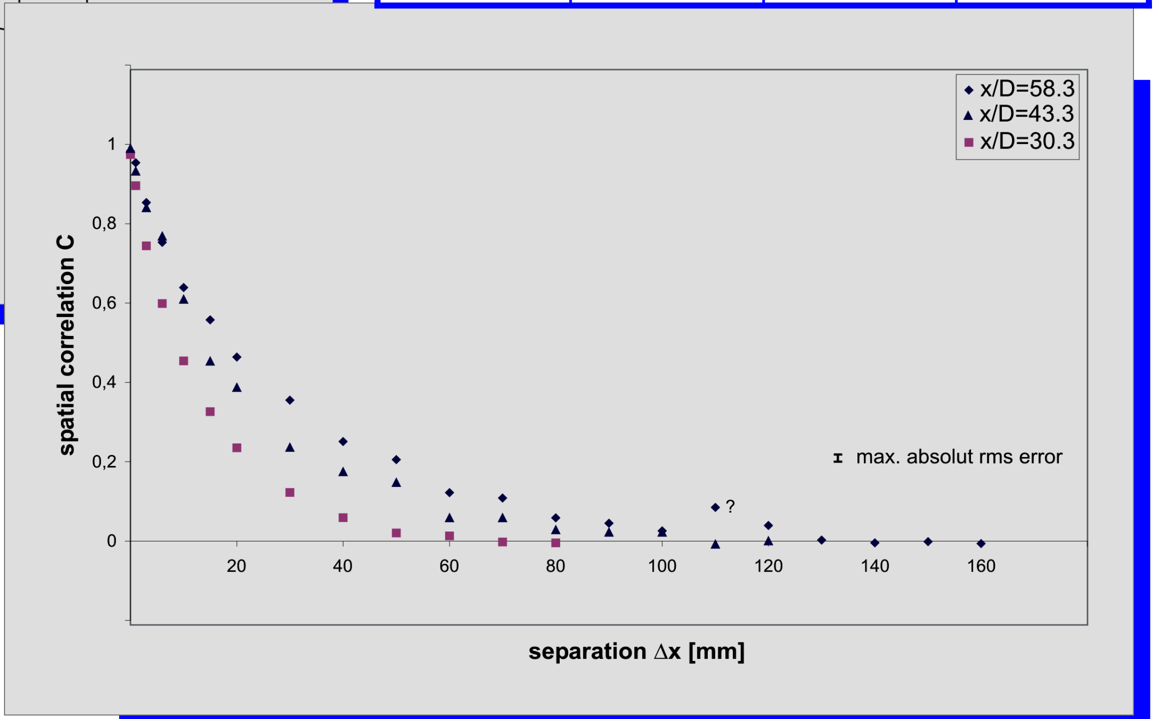
$\xi = \ln[(x-x_0)/D]$ and $\eta = r/\delta_{1/2}$

→ Scaled turbulence moments are **homogeneous** in ξ'-ξ and θ'-θ.

TWO-POINT CORRELATIONS



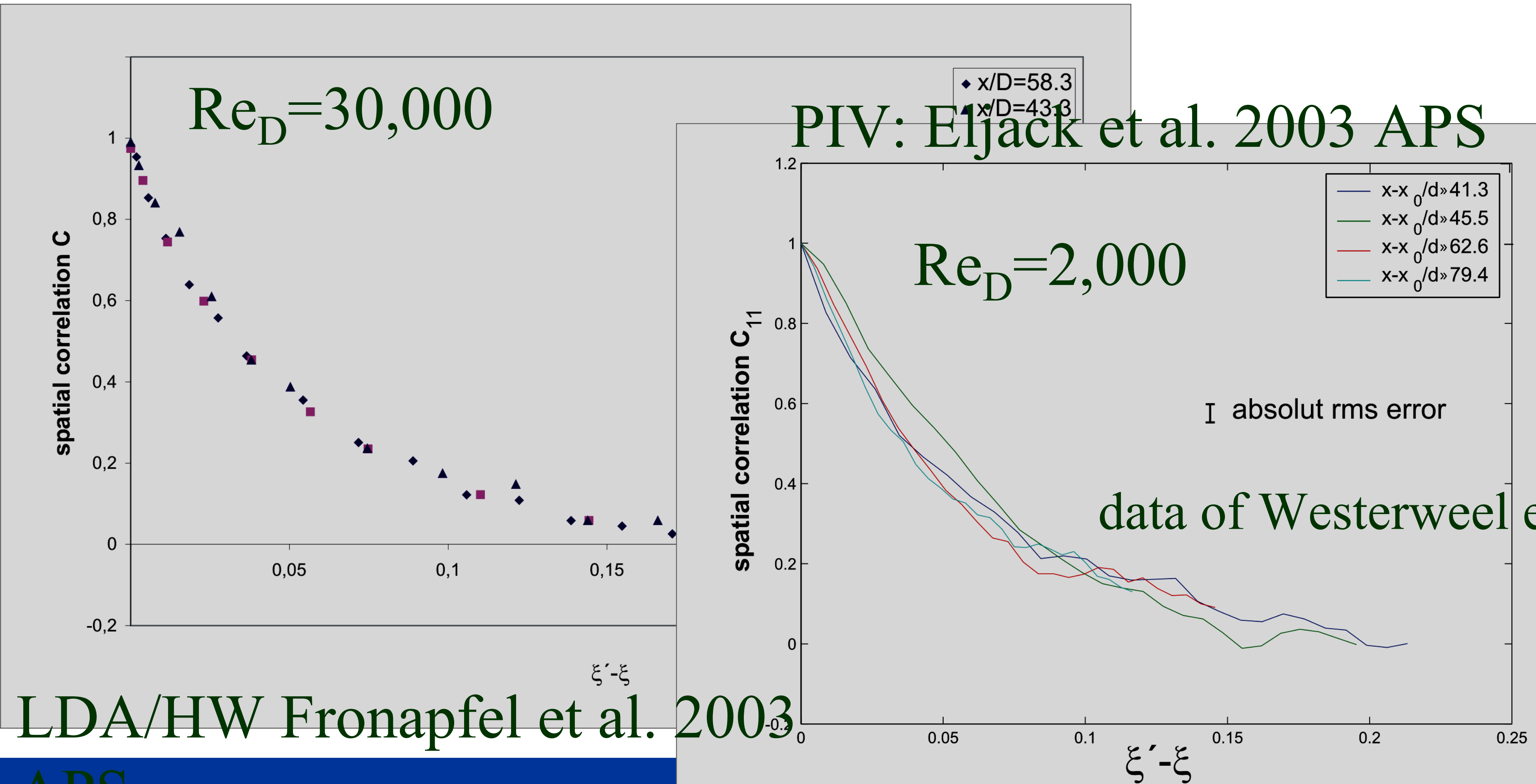
x/D	30.3	43.3	58.3
N	85714	17647	21429



LDA/HW Fronapfel et al.
2003 APS

Two-point correlations of streamwise velocity

$$\xi' - \xi = \ln[(x - x_o)/(x' - x_o)], \quad \xi \equiv \ln[(x - x_o)/D]$$



IMPLICATIONS

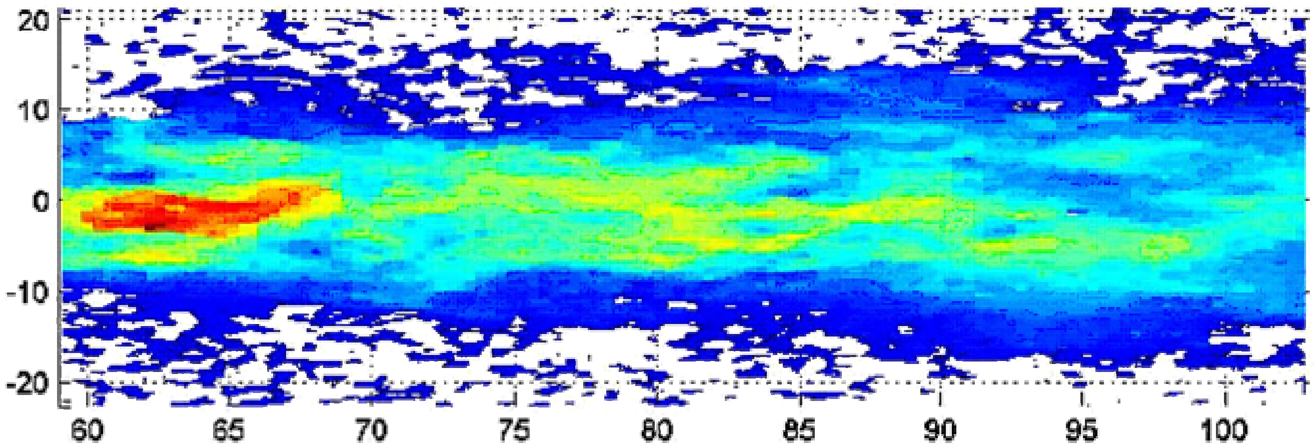
- POD application to inhomogenous flows of **infinite** extent (like streamwise direction of jet) is problematical since solutions depend on the domain chosen.

- BUT since the streamwise direction of the transformed jet is ***homogeneous in similarity variables*** …

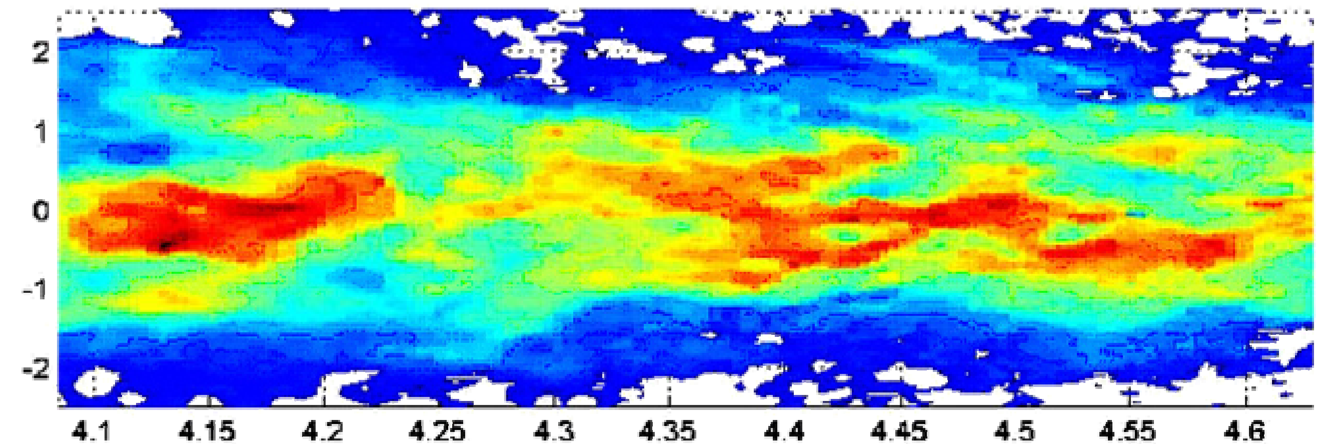
- solution to POD integral in the streamwise direction is Fourier modes in $\xi = \ln x / D$

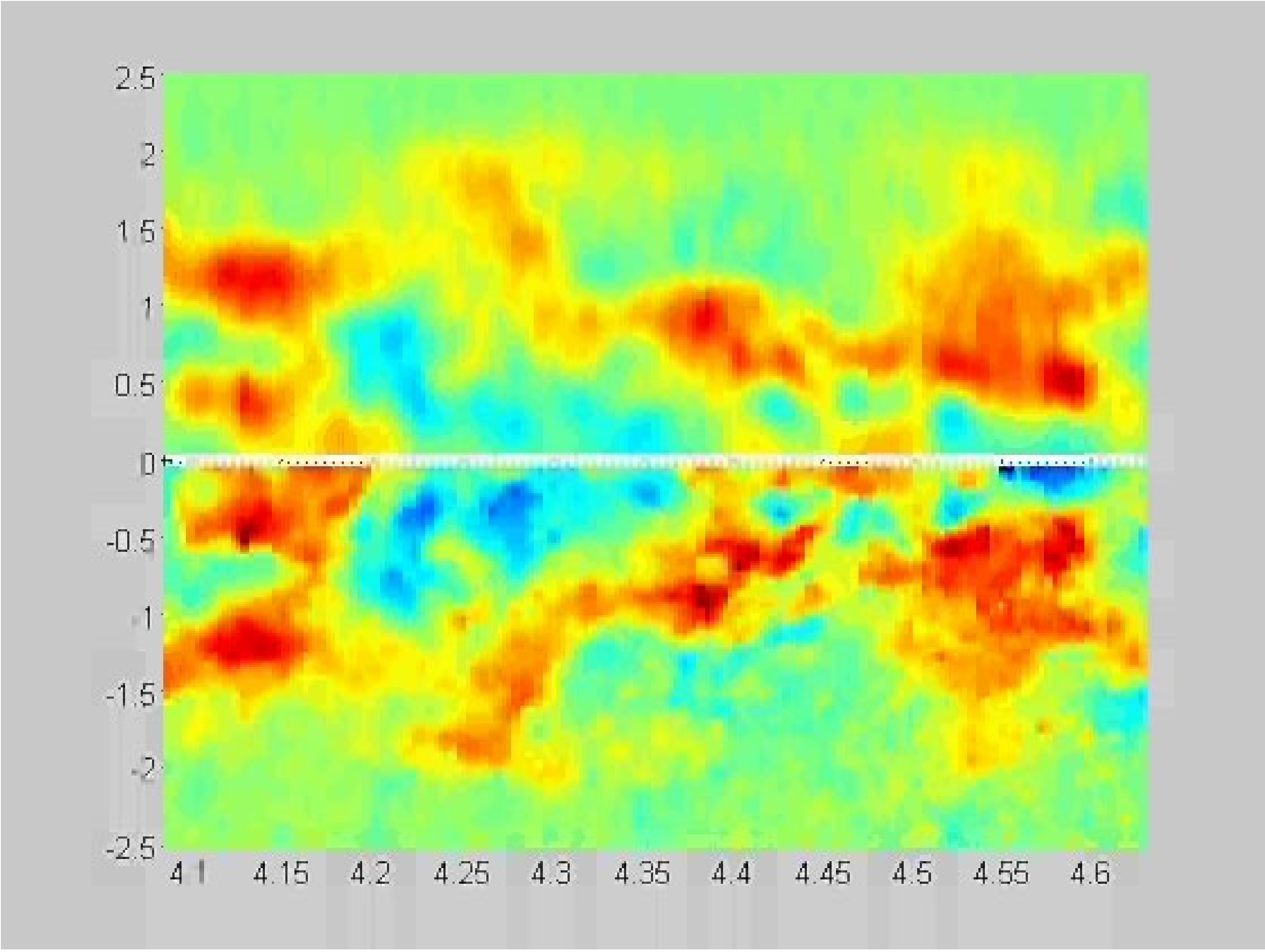
Jet flow field (PIV data of Westerweel et al. 2002)

Physical coordinates

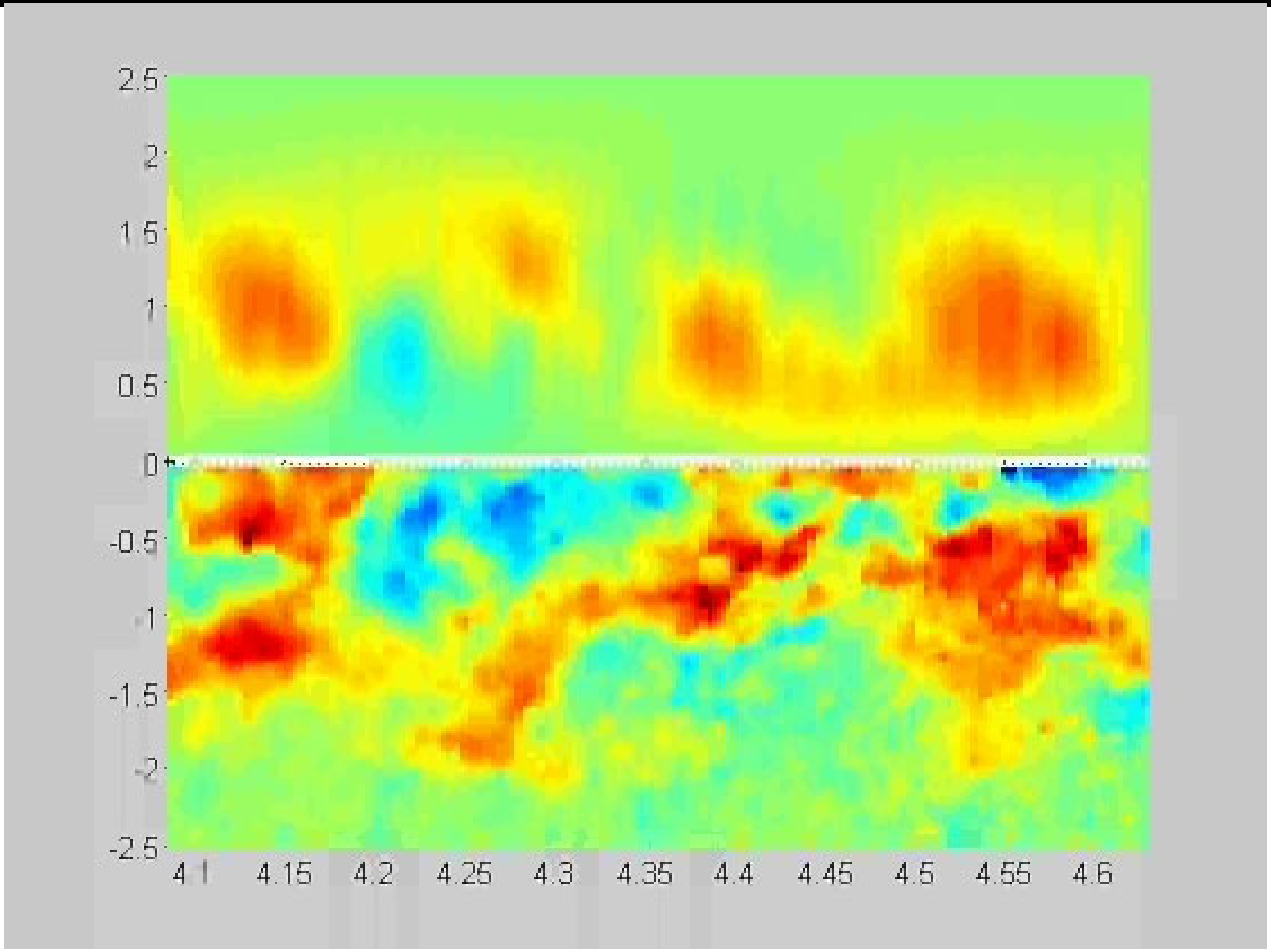


Similarity coordinates

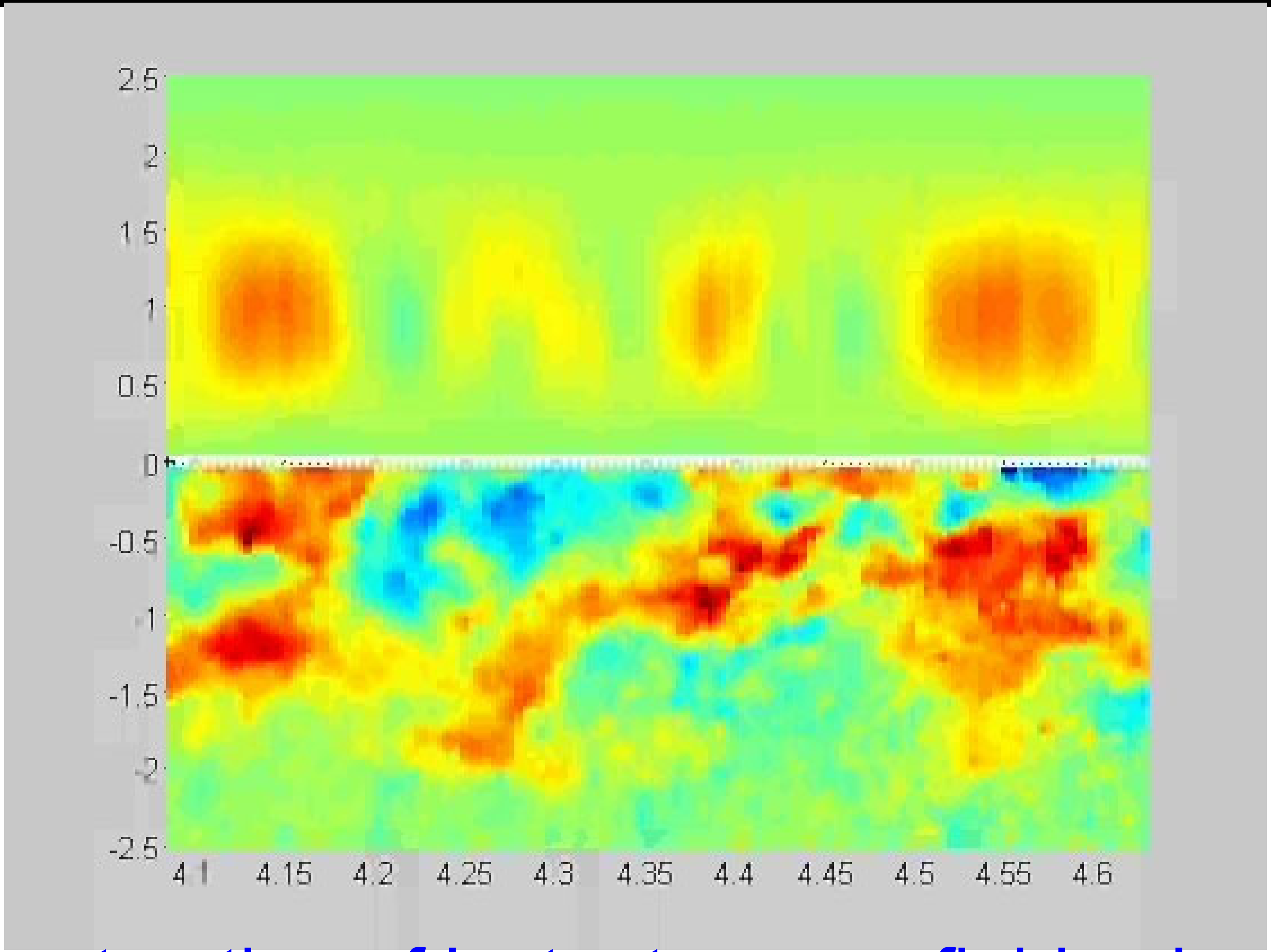




Reconstruction using POD modes 1-10

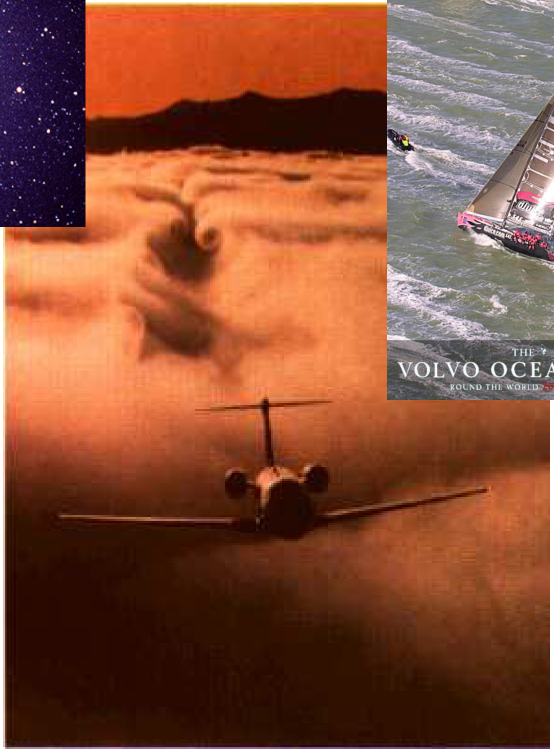


Reconstruction using radial POD modes 1-2 and all streamwise wavenumbers.



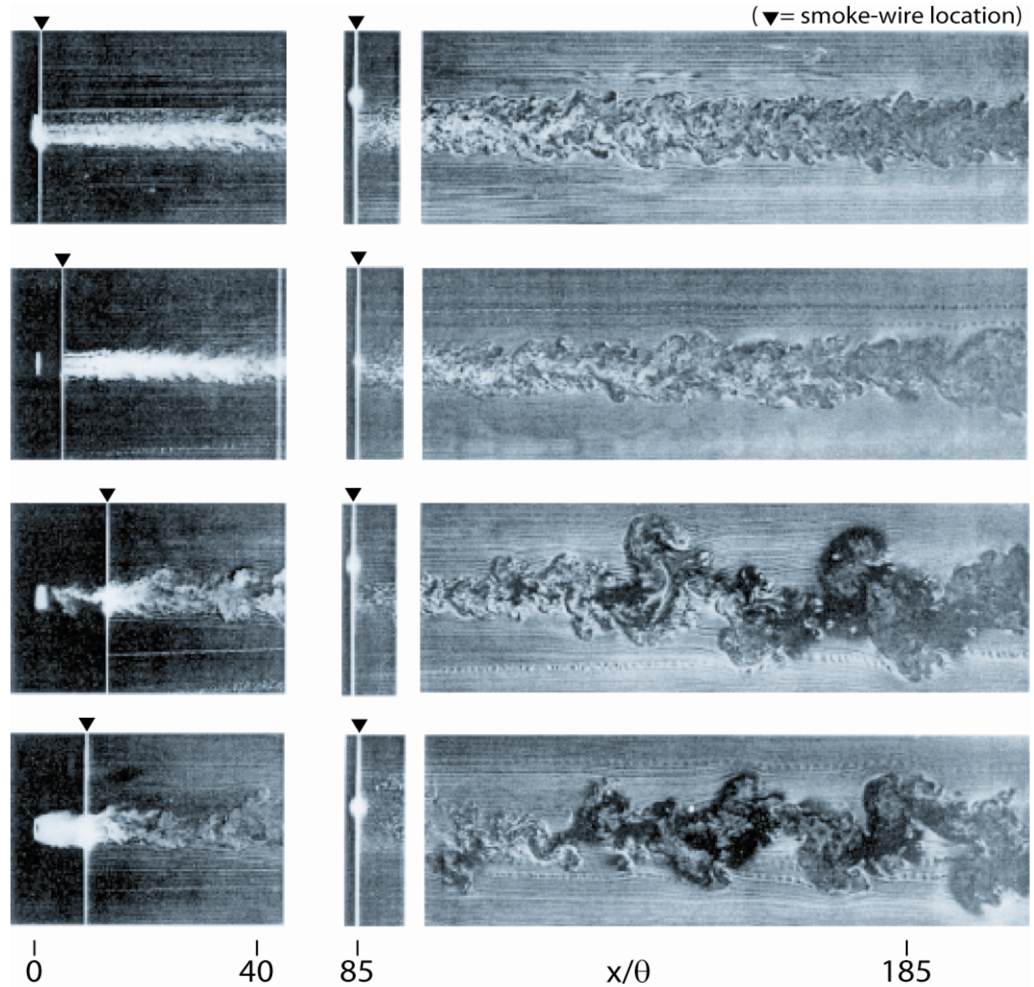
Reconstruction of instantaneous field using POD only radial mode 1 and all streamwise wavenumbers

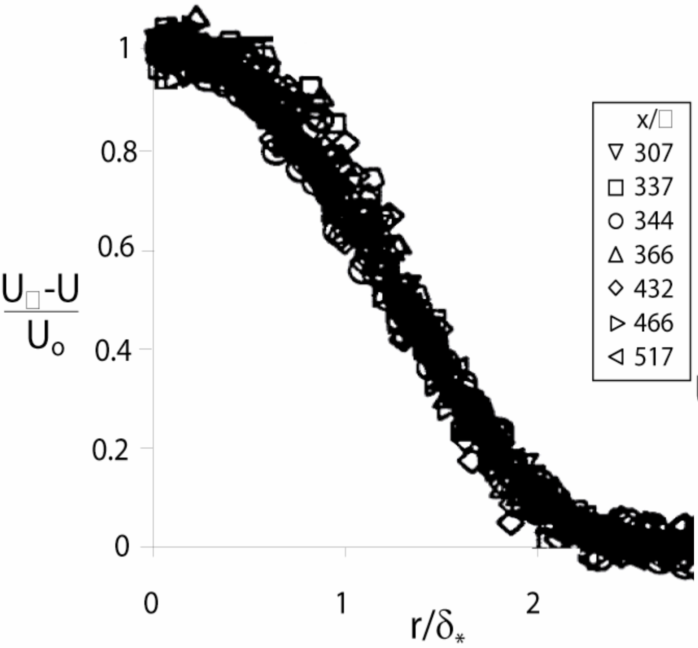
Now for a more difficult problem...: all wakes don't look alike either.



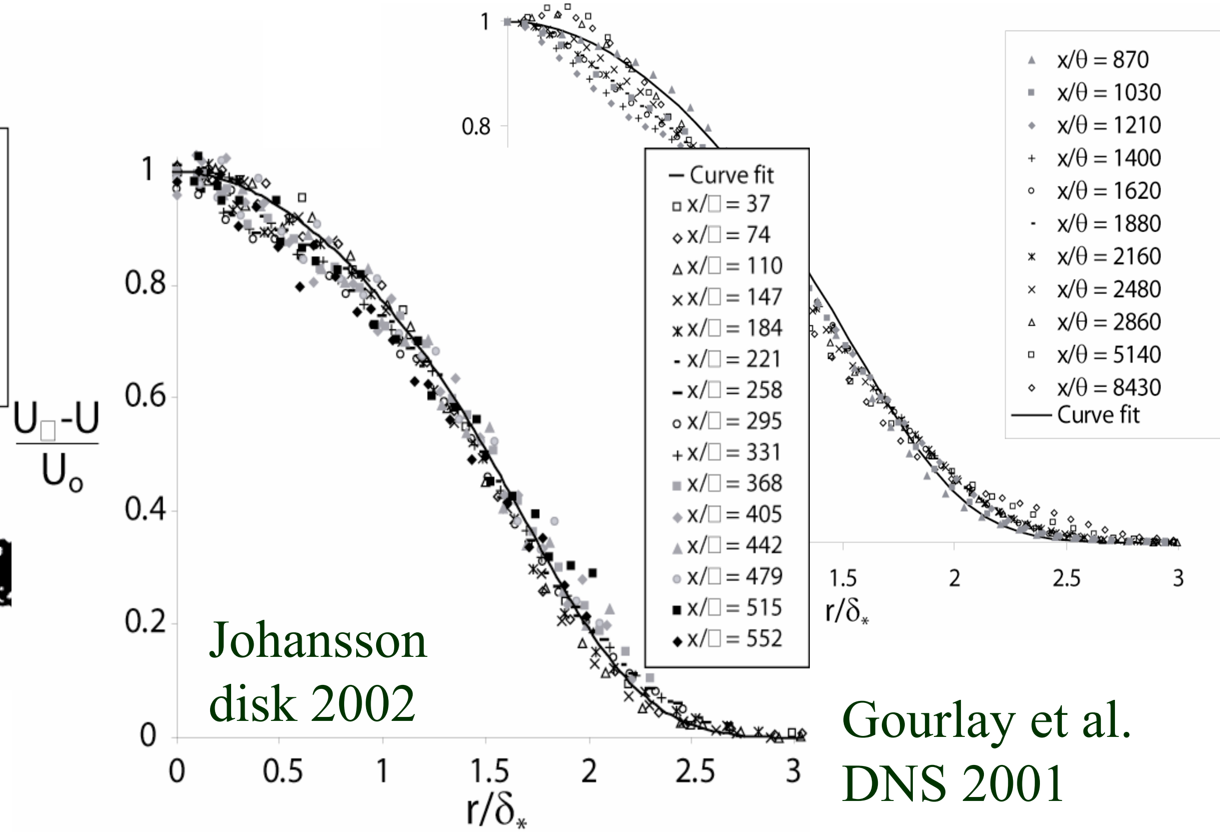
In fact, upstream (or initial) conditions are remembered far downstream!

Wakes behind four different axisymmetric generators ... suggesting strongly that far wakes retain dependence on initial conditions. (Cannon 1991).





Cannon porous disk 1991

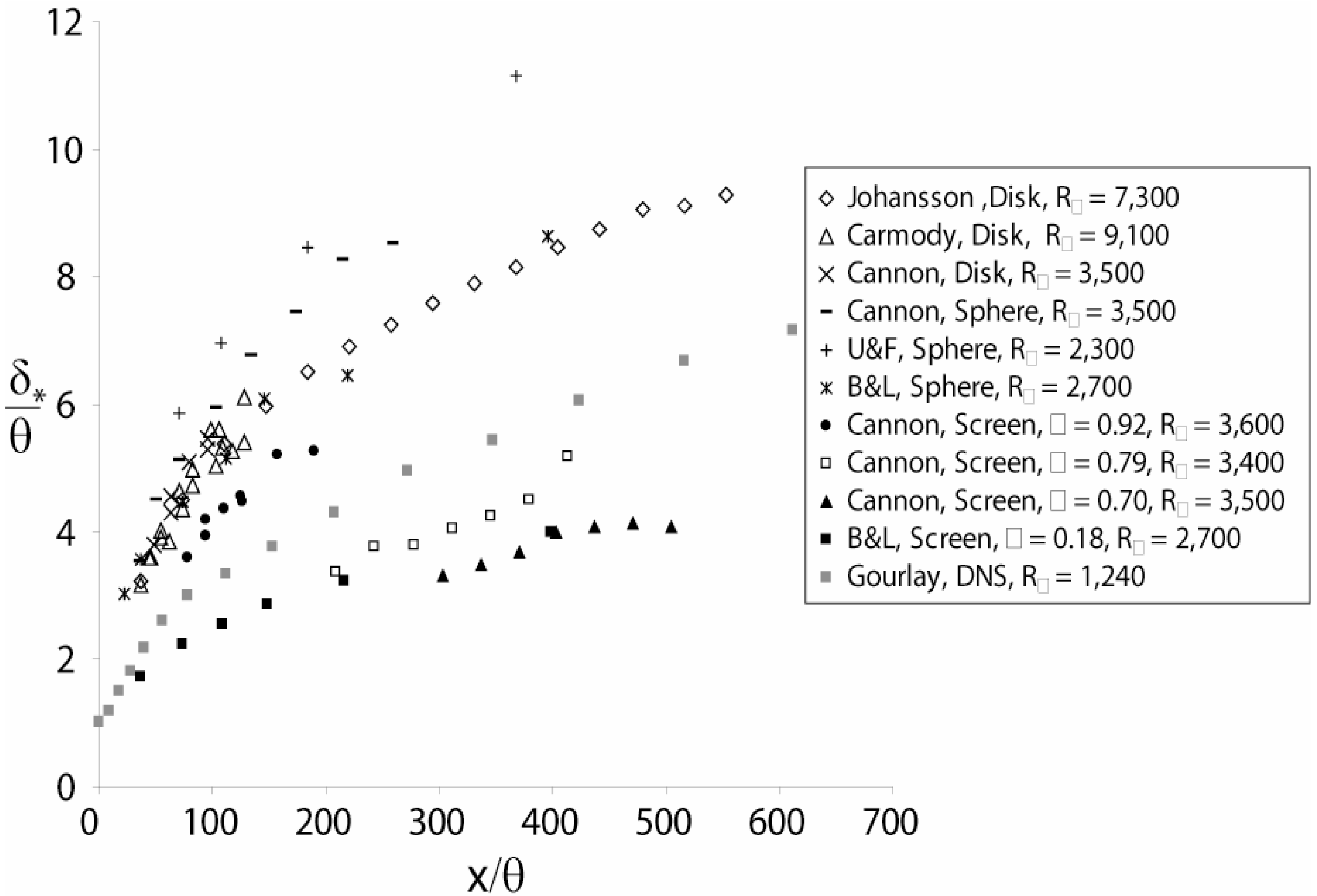


Johansson disk 2002

Gourlay et al. DNS 2001

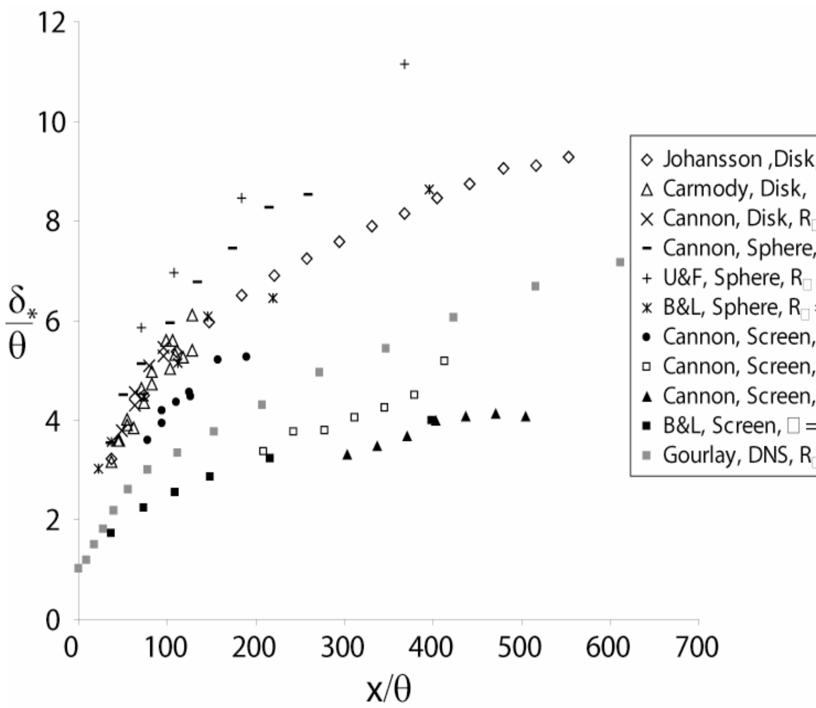
These high Reynolds number axisymmetric wake profiles are nearly identical - as **both** the equilibrium similarity and classical theories predict.

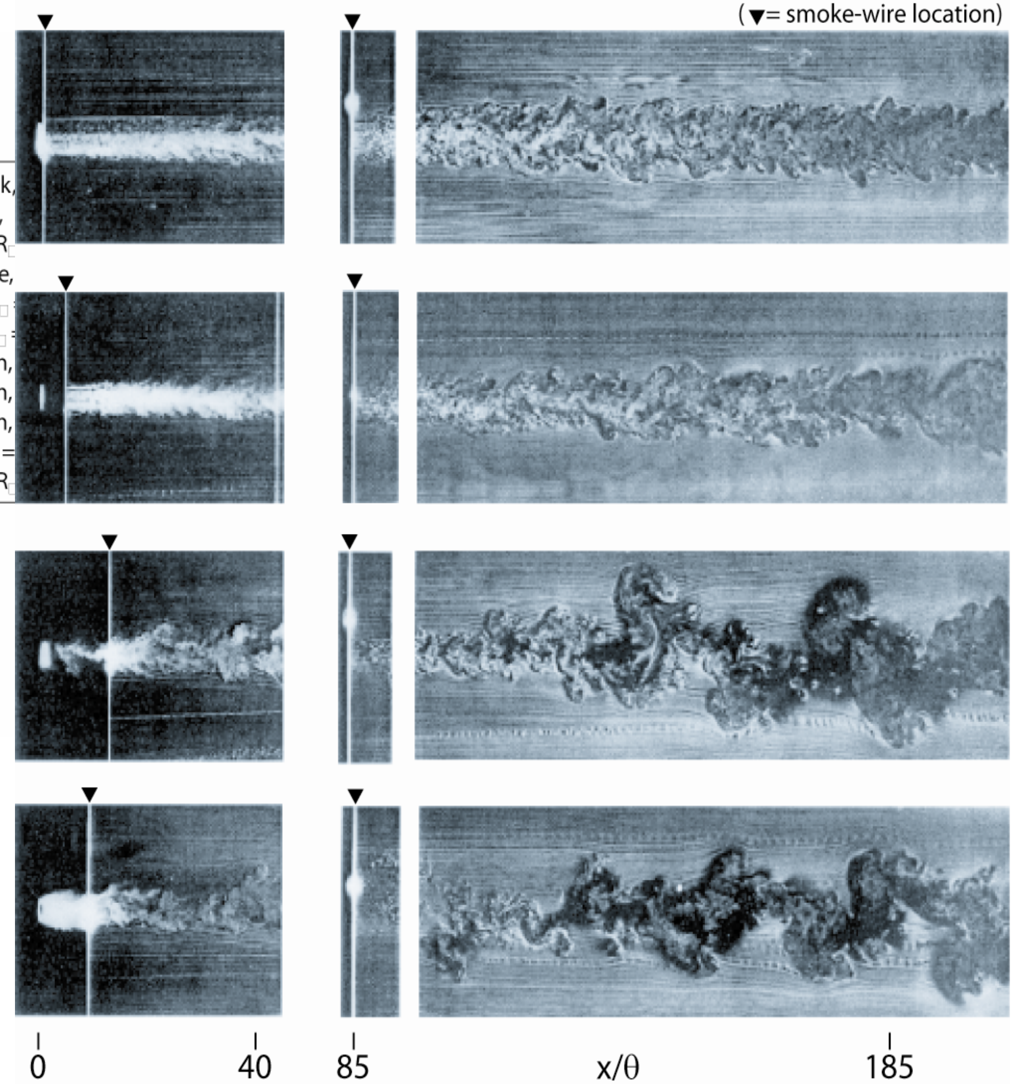
But only equilibrium similarity can explain this.



From P. B. V. Johansson et al. 2003 Phys. Fluids

Clearly structure does matter...





These pictures were telling us something after all.

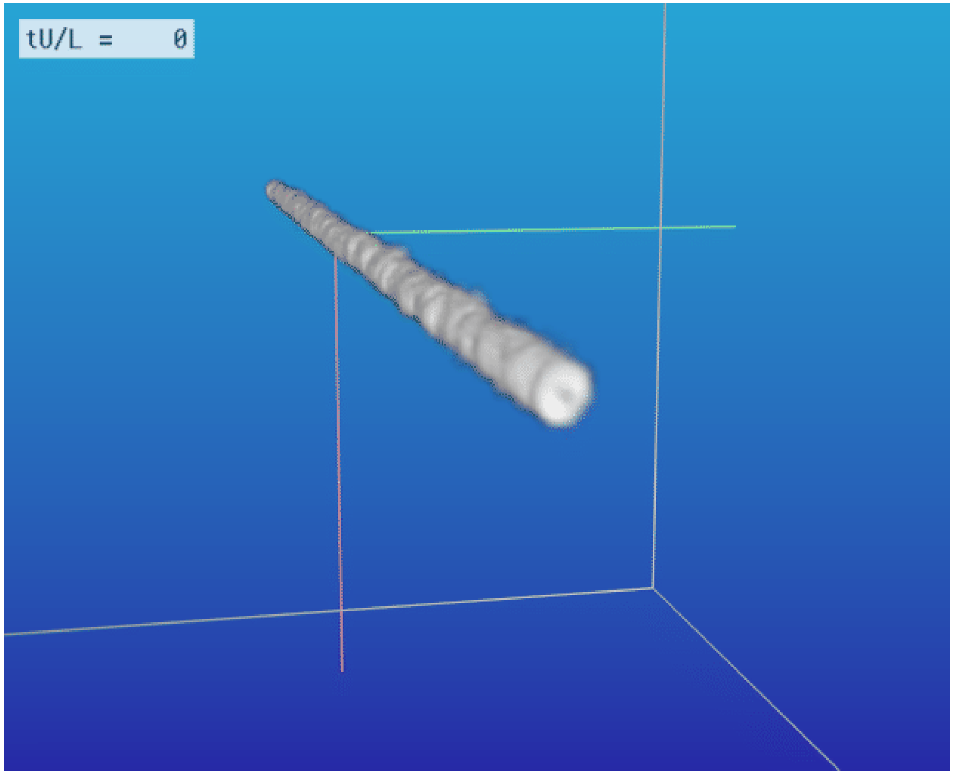
But there remains a problem for the axisymmetric wake...

- The local Reynolds number for the high Re solution ***decreases*** with downstream distance, i.e.,

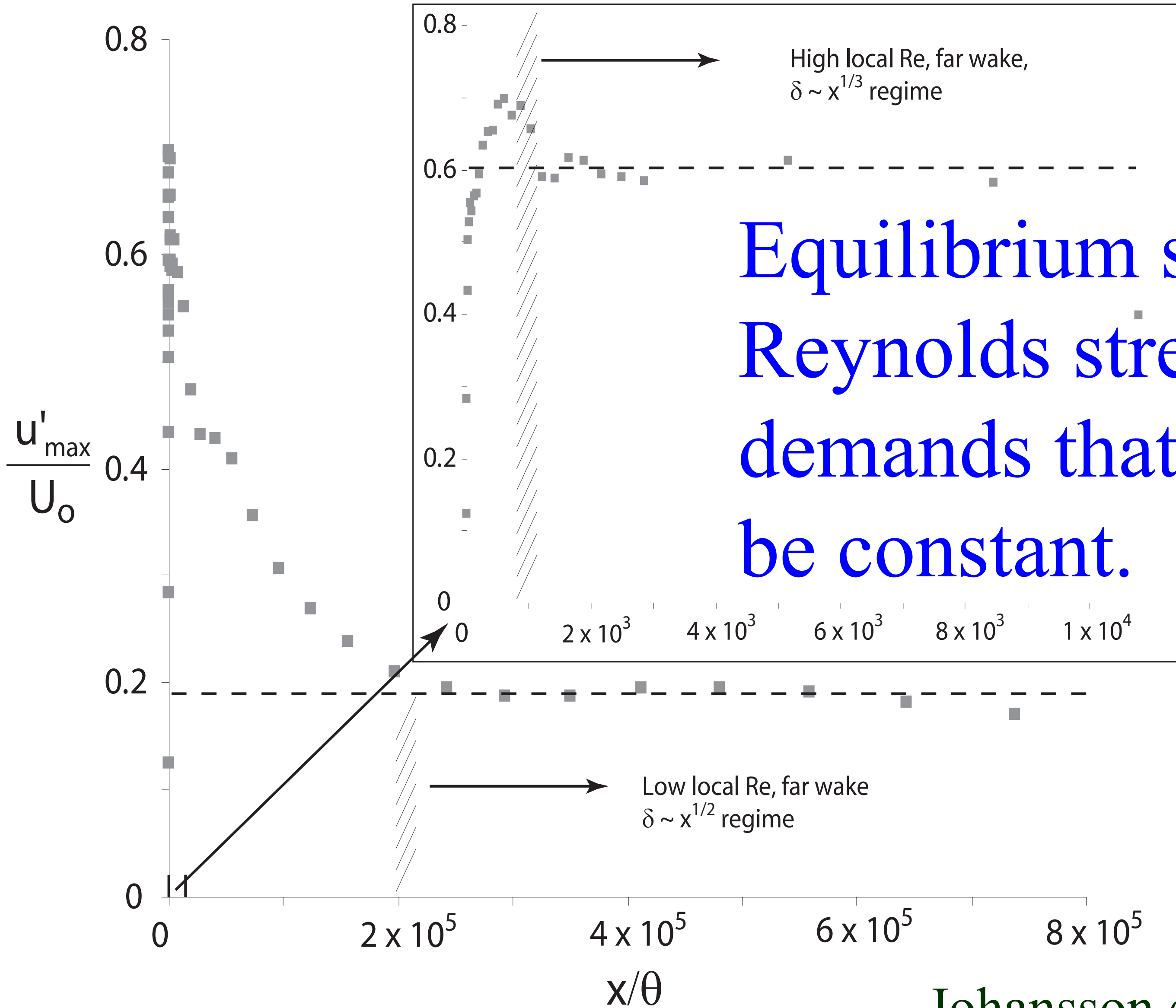
$$\mathrm{Re} = (U_\infty - U_{cl})\delta_* / \nu \quad \propto \quad (x - x_o)^{-1/3}$$

- Thus our ***infinite*** Re equations slowly become invalid– if they ever were valid in the first place.

This animation gives a clue. Note how the vortex cores thicken as wake evolves downstream.



DNS of axisymmetric wake: wake generator moves from right to left. Gourlay et al. (2001)

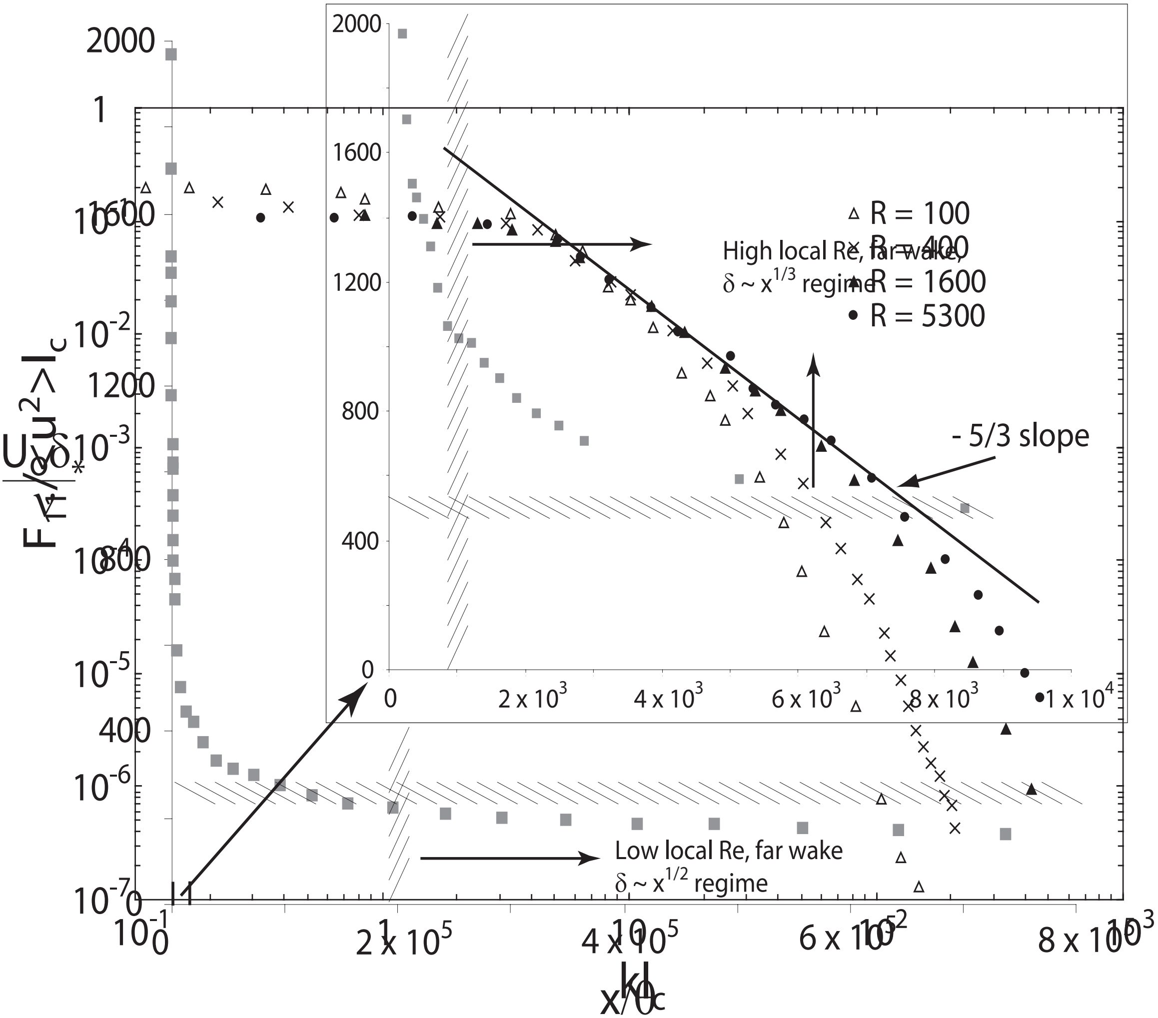


Equilibrium similarity of Reynolds stress equations demands that u'_{max} / U_o be constant.

Johansson et al 2003 Phys. Flds

Local Reynolds number diminishes with distance.

Velocity Spectra of Uberoi and Freymuth 1970

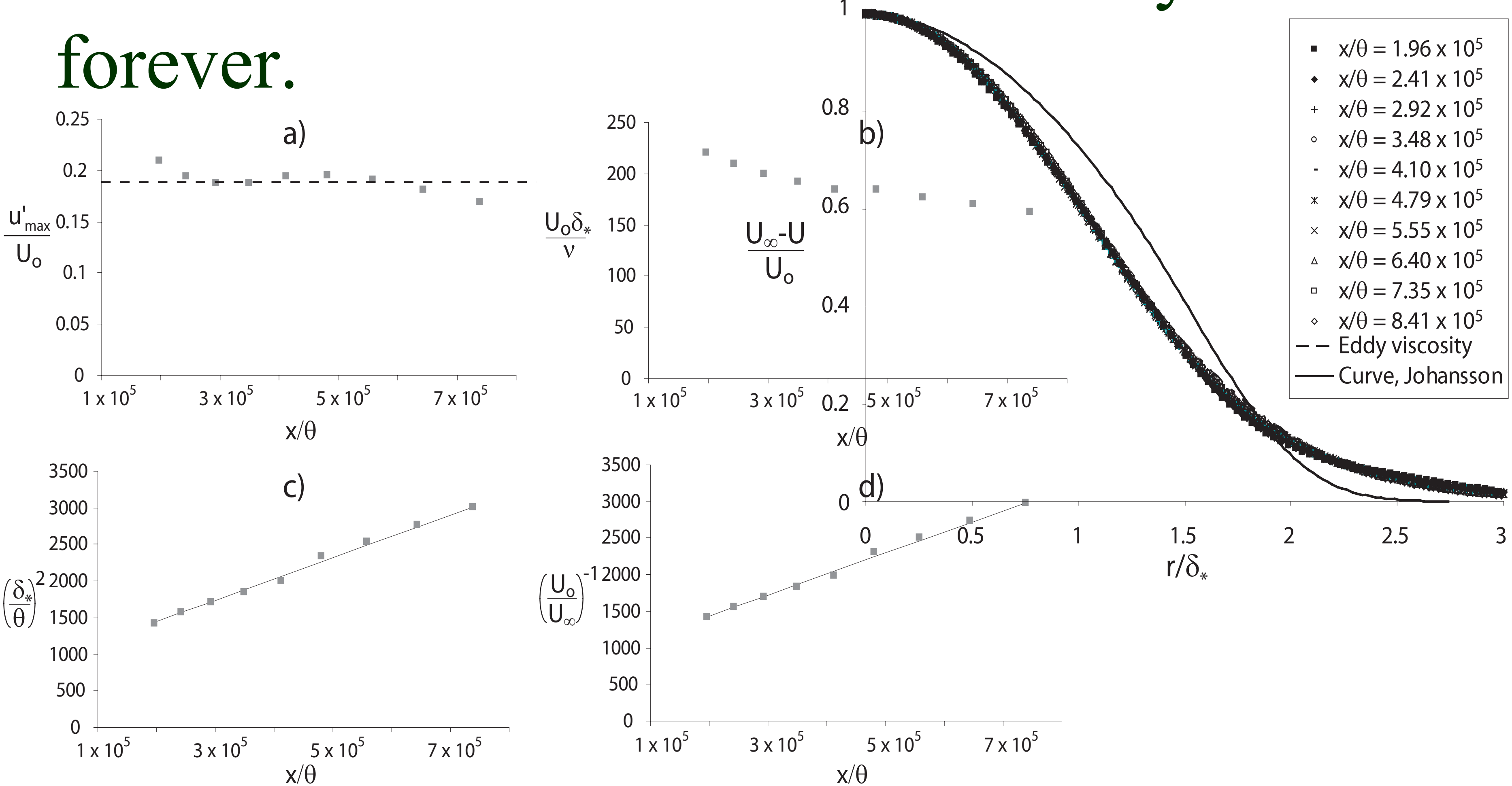


Johansson et al 2003 Phys. Flds.

- Deviations from hig
Re solution occur
quite early – not nea
turbulence Re of
unity, but several
hundred!

- NOT
laminarization,
but low Re
turbulence.

There is a second similarity state... to which the flow evolves – and stays forever.



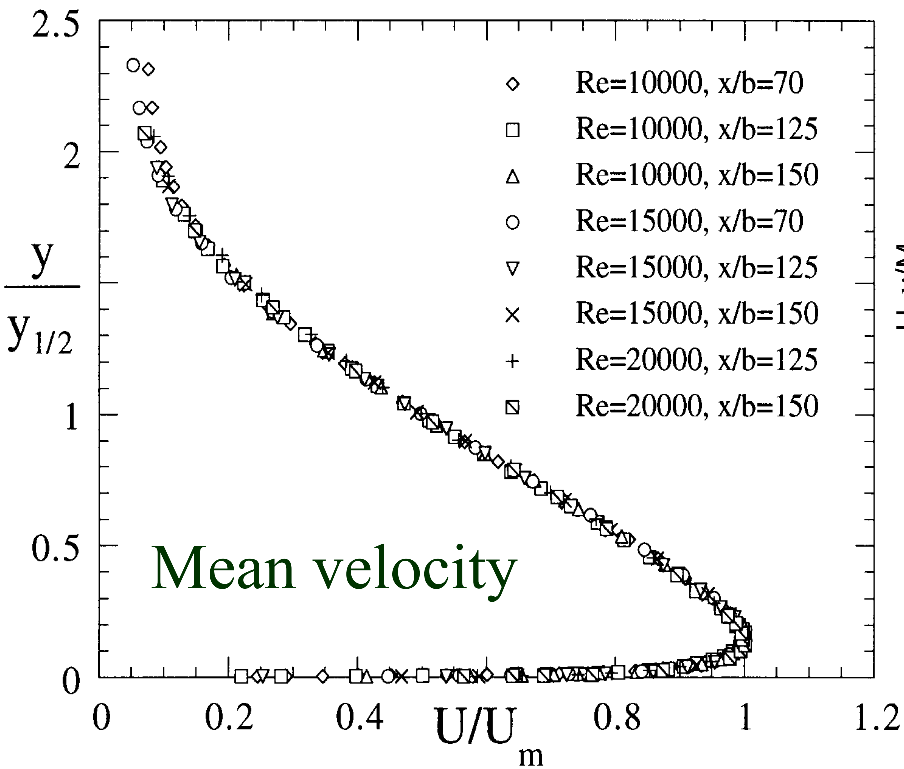
Other examples of evolving flows

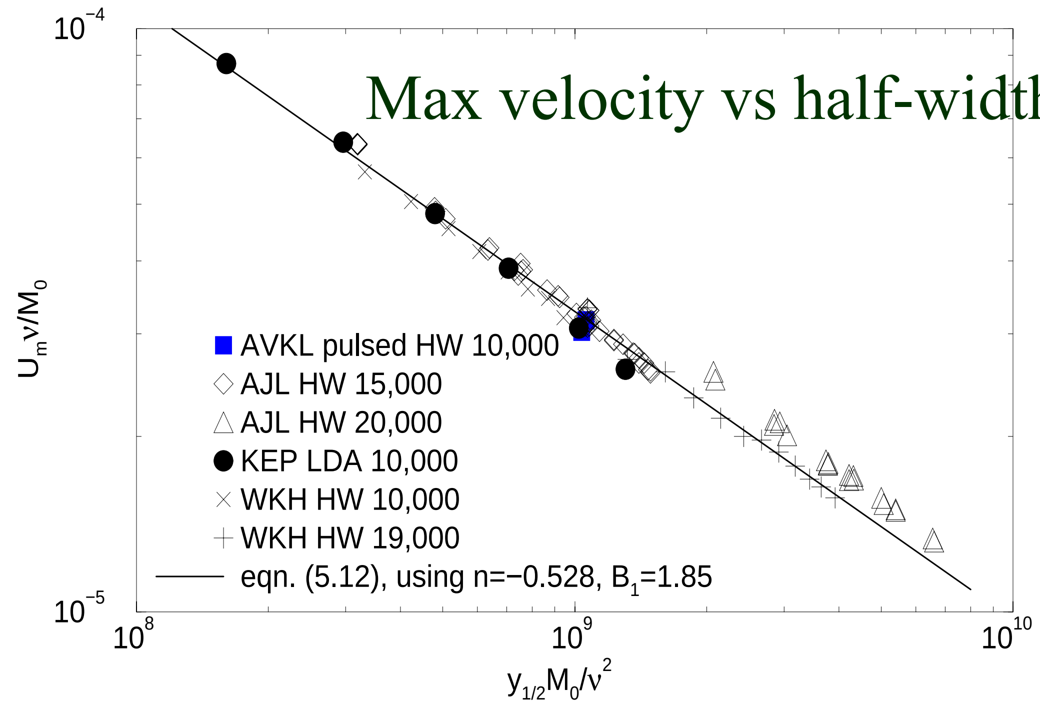
- The outer part (90% plus) of turbulent boundary layer flows do this in reverse.

- As the Reynolds number increases downstream, they evolve toward the infinite Re solution (which is the only equilibrium similarity solution).

- This can be quite frustrating to those who want simple scaling laws for finite Re experiments or DNS.

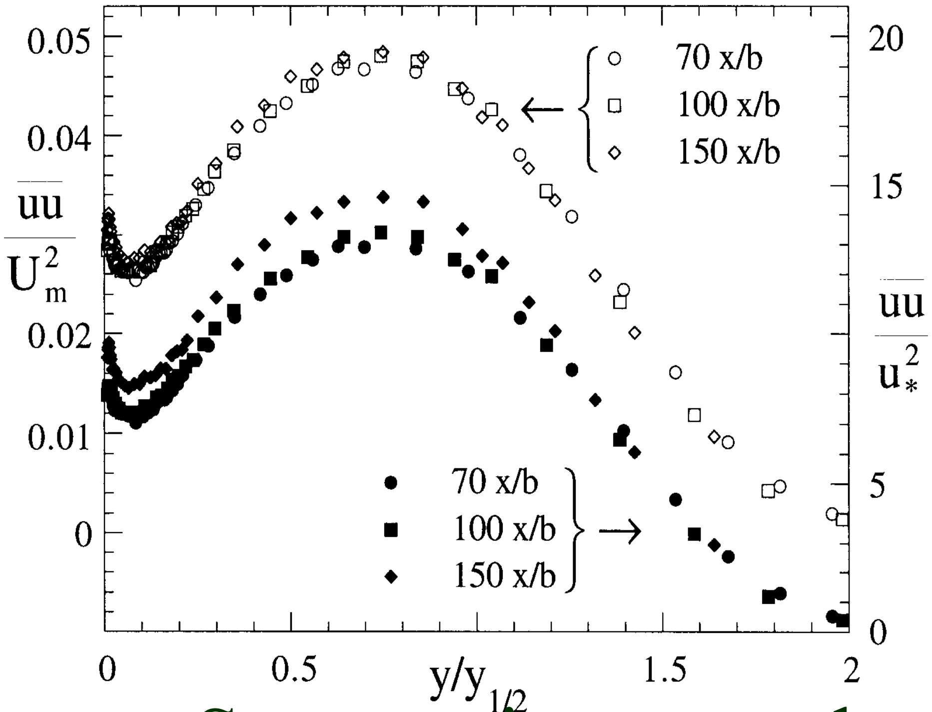
Even so, sometimes the asymptotic equilibrium similarity solutions can work quite well – especially if driven by a strong shear, imposed strain rate or pressure gradient.





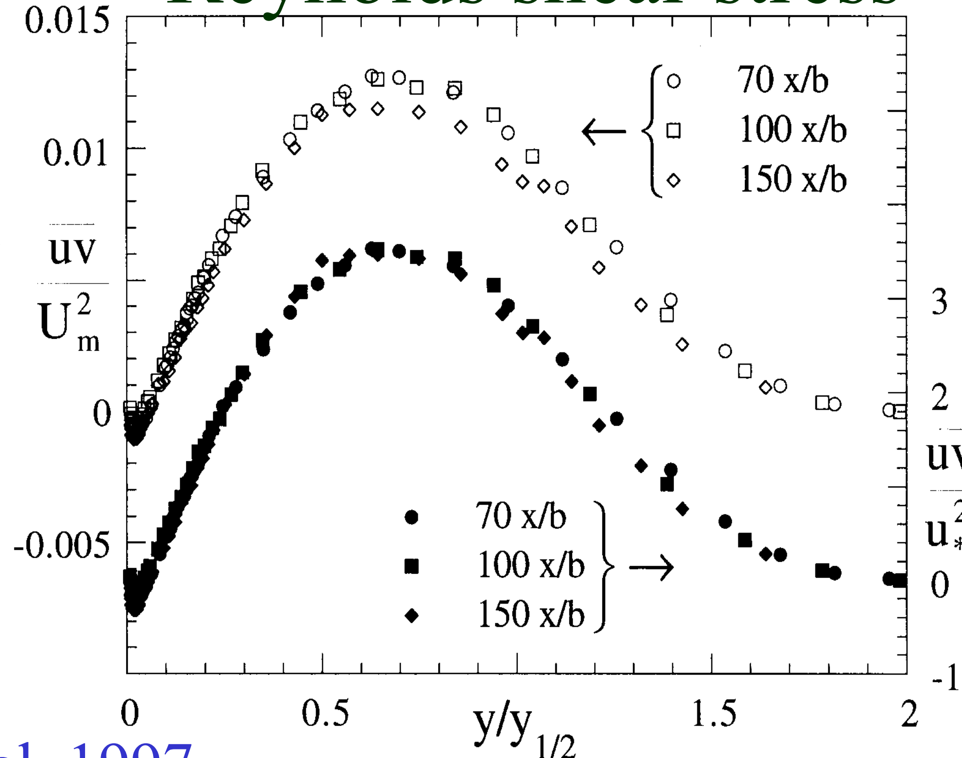
Plane walljet, George et al. 2000 JFM

Result of using near-asympototics to 'match' outer equilibrium similarity solution to inner is that normal stresses scale differently than shear. $U_\infty^2 d\delta / dx \rightarrow u_*^2$



Streamwise normal

Reynolds shear stress



Plane walljet data Abrahamsson et al. 1997

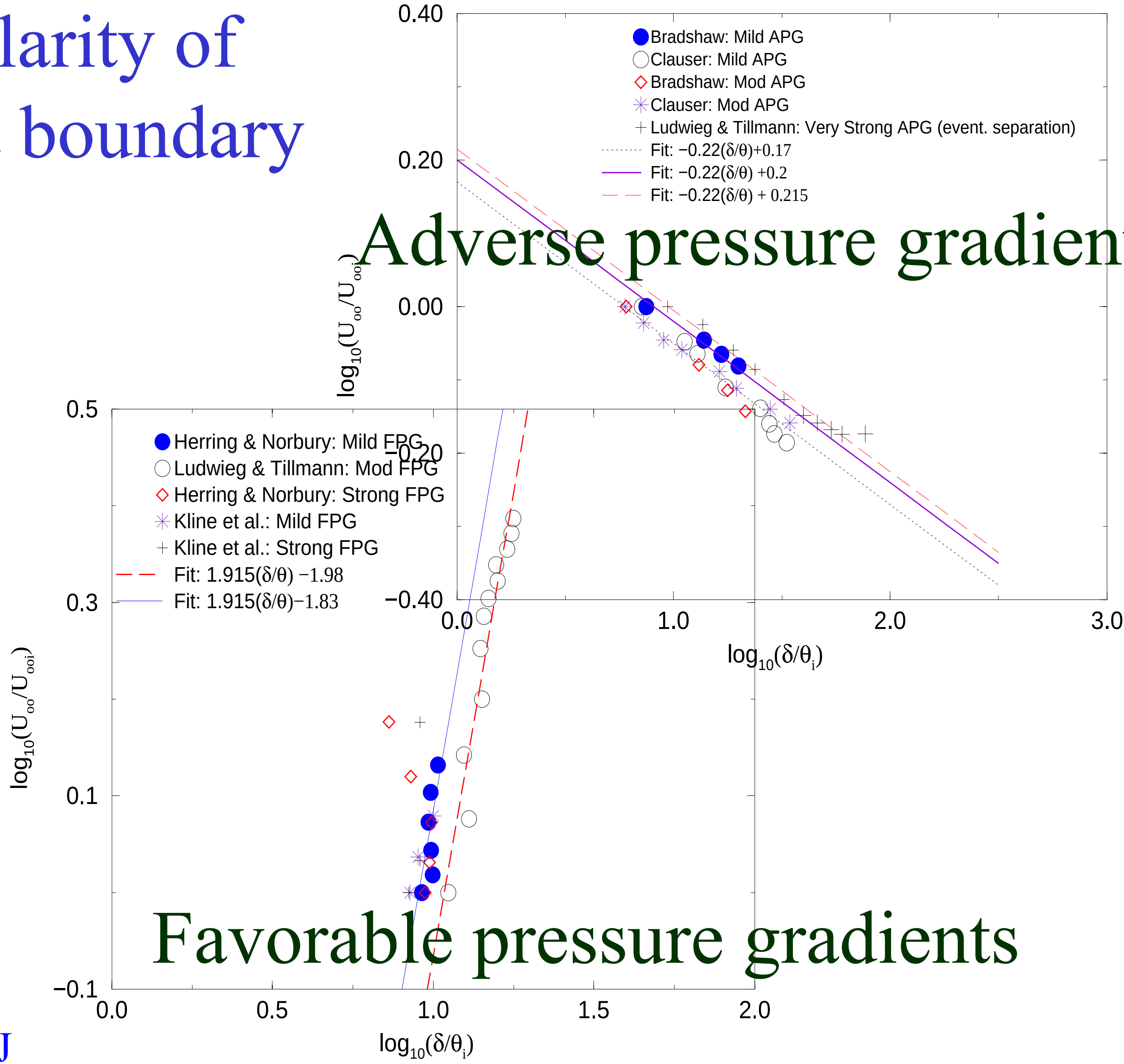
Equilibrium similarity of pressure gradient boundary layer implies

$$\delta \propto U_\infty^{1/\Lambda}, \quad \Lambda \neq 0$$

$$\Lambda \equiv -\frac{\delta}{\rho d\delta/dx}\frac{dP_\infty}{dx}$$

$$= const$$

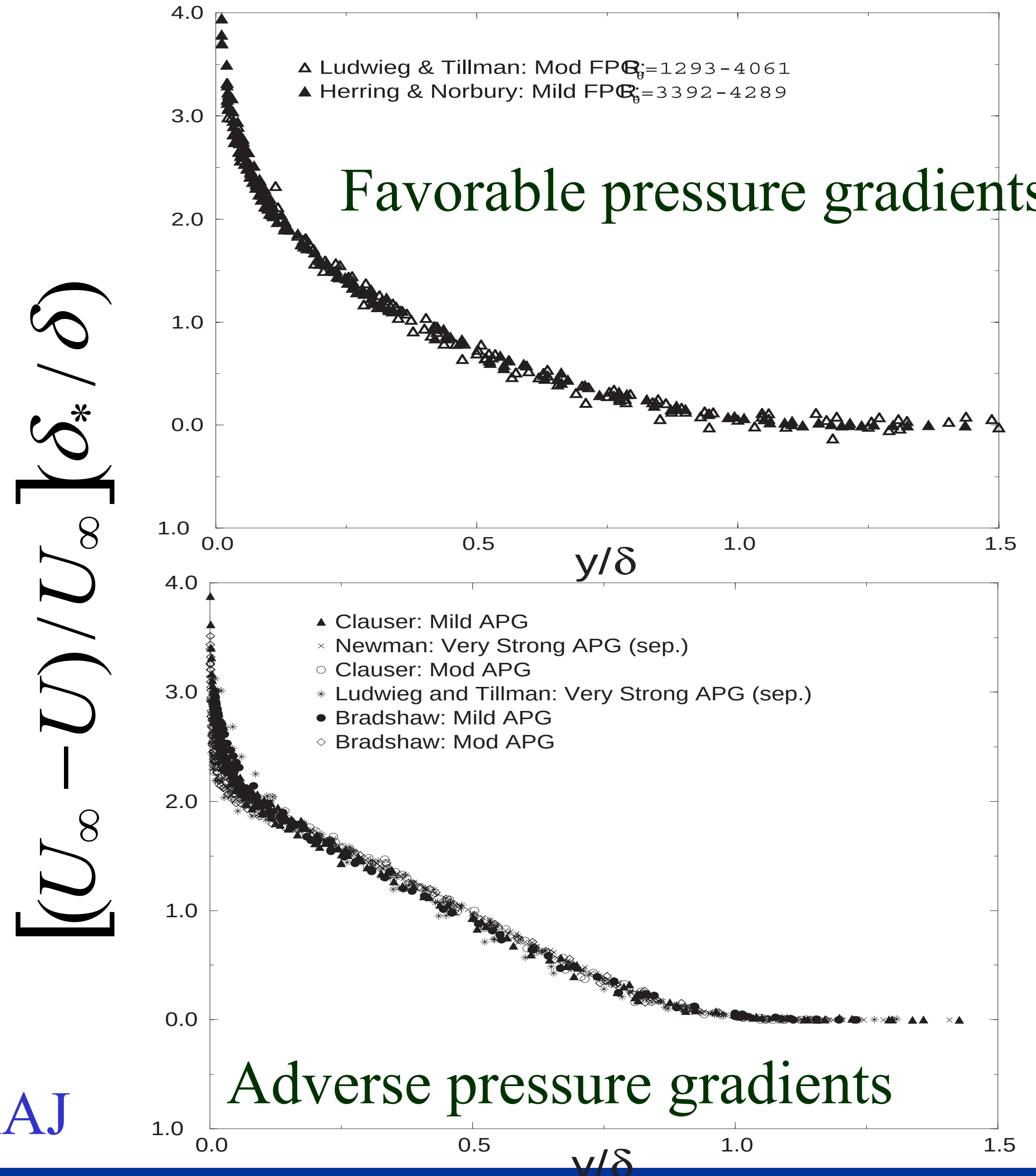
$$-\Lambda(2f'_{op\infty} + f^2_{op\infty}) + (\Lambda - 1)\bar{y}f'_{op\infty}$$
$$+(\Lambda - 1)f'_{op\infty}\int_0^{\bar{y}} f_{op\infty}(\hat{y})d\hat{y} = \tilde{r}'_{op\infty}$$



Castillo and George 2001 AIAAJ

The velocity profiles collapse pretty well too in equilibrium similarity variables – especially with the Smits-Zagarola scaling to remove the Reynolds number and upstream dependence.

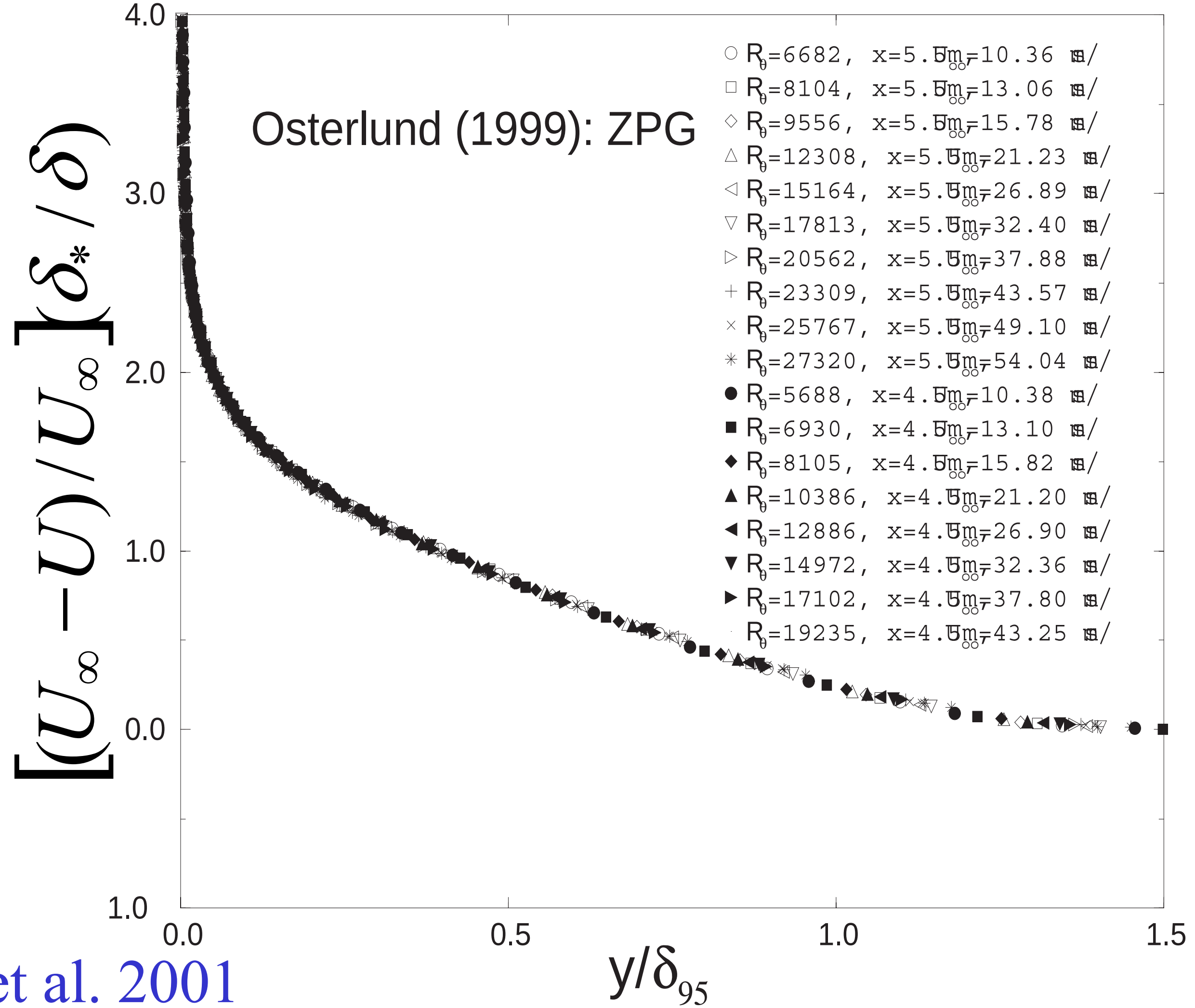
$$-\Lambda(2f'_{op\infty} + f^{2}_{op\infty}) + (\Lambda-1)\bar{y}f'_{op\infty}$$
$$+(\Lambda-1)f'_{op\infty}\int_0^{\bar{y}} f_{op\infty}(\hat{y})\,d\hat{y} = \tilde{r}'_{op\infty}$$



Castillo and George 2001 AIAAJ

... and even for zero pressure gradient.

$$-\bar{y}f_{o\infty}' - f_{o\infty}' \int_0^{\bar{y}} f_{o\infty}(\hat{y})d\hat{y} = \tilde{r}_{o\infty}'$$



Wosnik 2000, Castillo et al. 2001

Obviously which terms come into play and where is very important.?

- Best example of this are the strained wake DNS of Rogers (2003 JFM).
- **Ten** different time-dependent equilibrium similarity solutions were identified, depending on the orientation and magnitude of the applied strain rate.
- Equilibrium similarity solutions appear to behave as very powerful 'attractors'.

Some problems remain for isotropic decaying turbulence:....

- Spectral energy equation

$$\frac{\partial E}{\partial t} = T - 2\nu k^2 E$$

- E(k,t) is three dimensional energy spectrum function (averaged over spherical shells of radius k).
- T(k,t) is the non-linear spectral transfer.

$$\frac{\partial E}{\partial t} = T - 2\nu k^2 E$$

George 1992 Phys. Fluids (G92) seeks single length scale similarity solutions of type:

$$E(k,t) = E_s(t)F(\bar{k})$$

$$T(k,t) = T_s(t)G(\bar{k})$$

$$\bar{k} = kL(t)$$

Spectral Similarity Equation

$$\left[\frac{L^2 \dot{E}}{\nu E_s}\right] F + \left[\frac{L \dot{L}}{\nu}\right] \bar{k} F' = \left[\frac{L^2 T}{\nu E_s}\right] G + [2] \bar{k}^2 F$$

temporal decay

non-linear spectral transfer

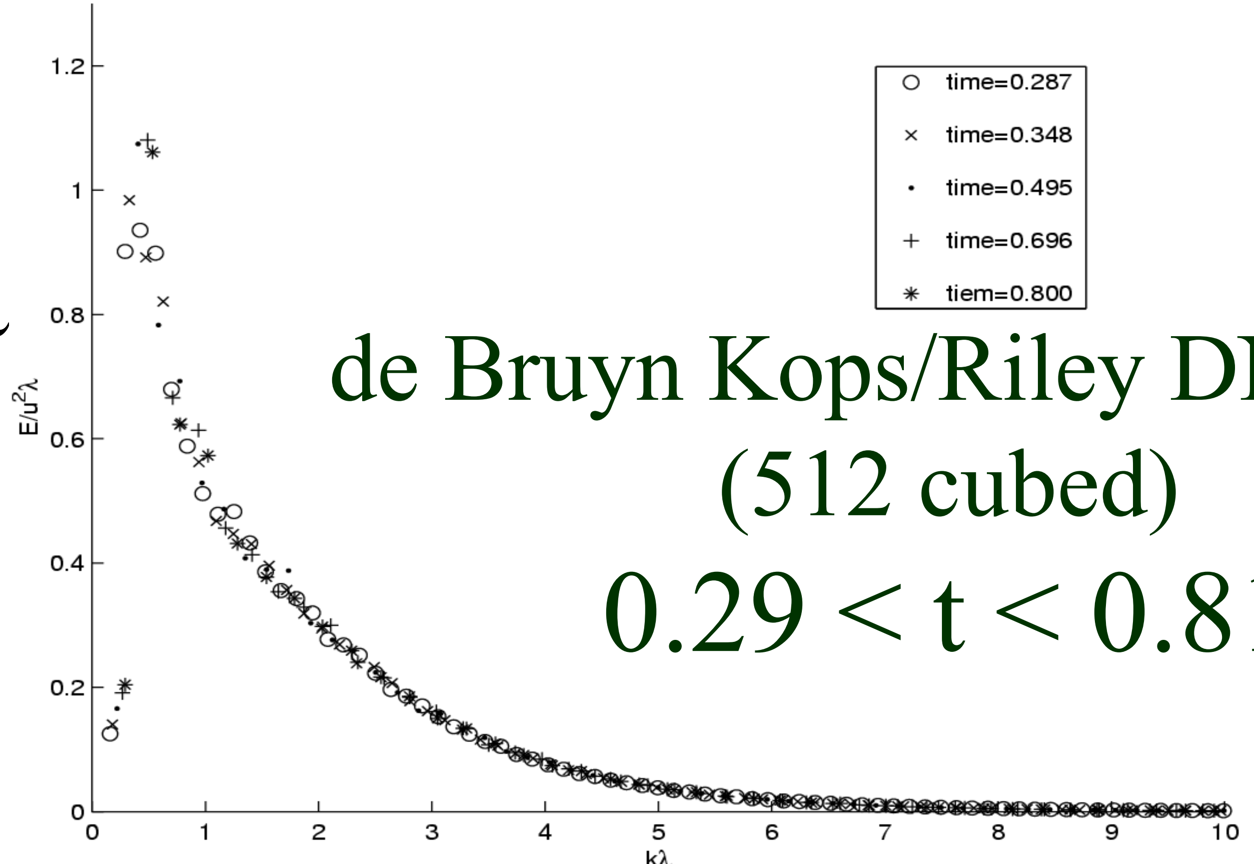
viscous dissipation

Equilibrium similarity hypothesis:

All of the terms in square brackets of the equation must evolve with time in exactly the same way (unless they are identically zero). There are no further assumptions.

- Theory implies spectra collapse (for fixed initial conditions) when plotted as:

$F = E/u^2\lambda$



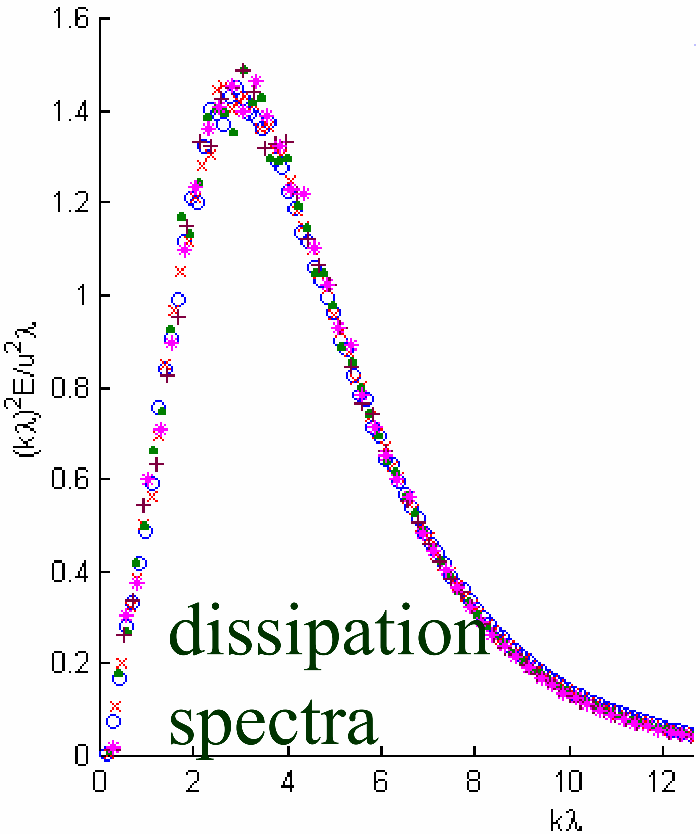
de Bruyn Kops/Riley DNS data (512 cubed)

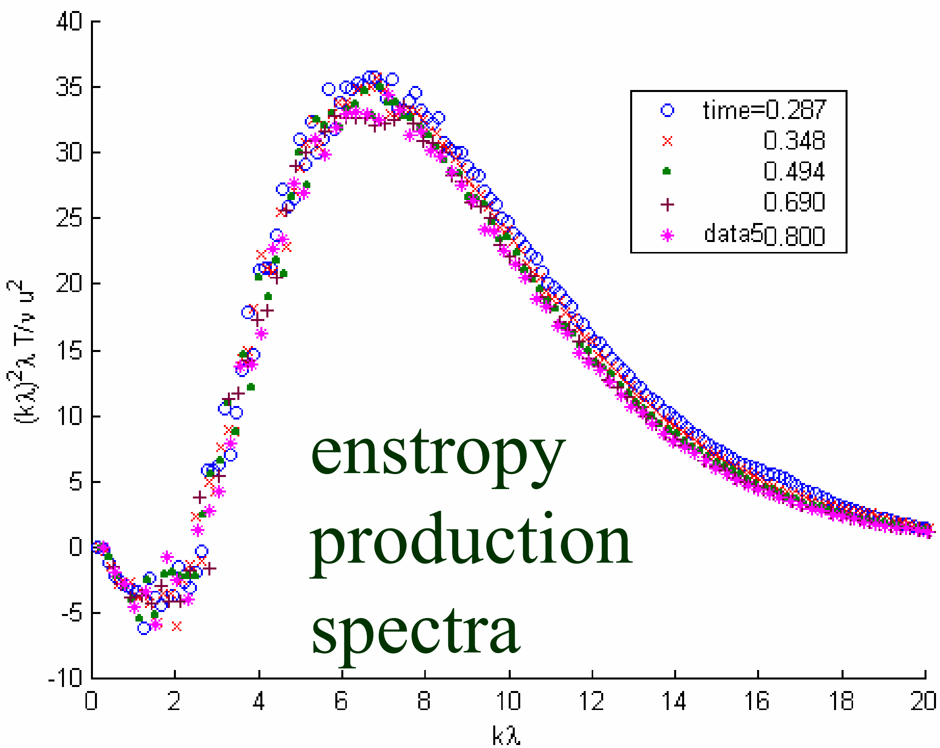
$0.29 < t < 0.81$

$k\lambda$

George and Wang 2002

de Bruyn Kops/Riley DNS data (512 cubed) 0.29 < t < 0.81





Wang and George 2002 JFM

• G92 equilibrium similarity theory deduces that the non-linear transfer is related to spectrum by:

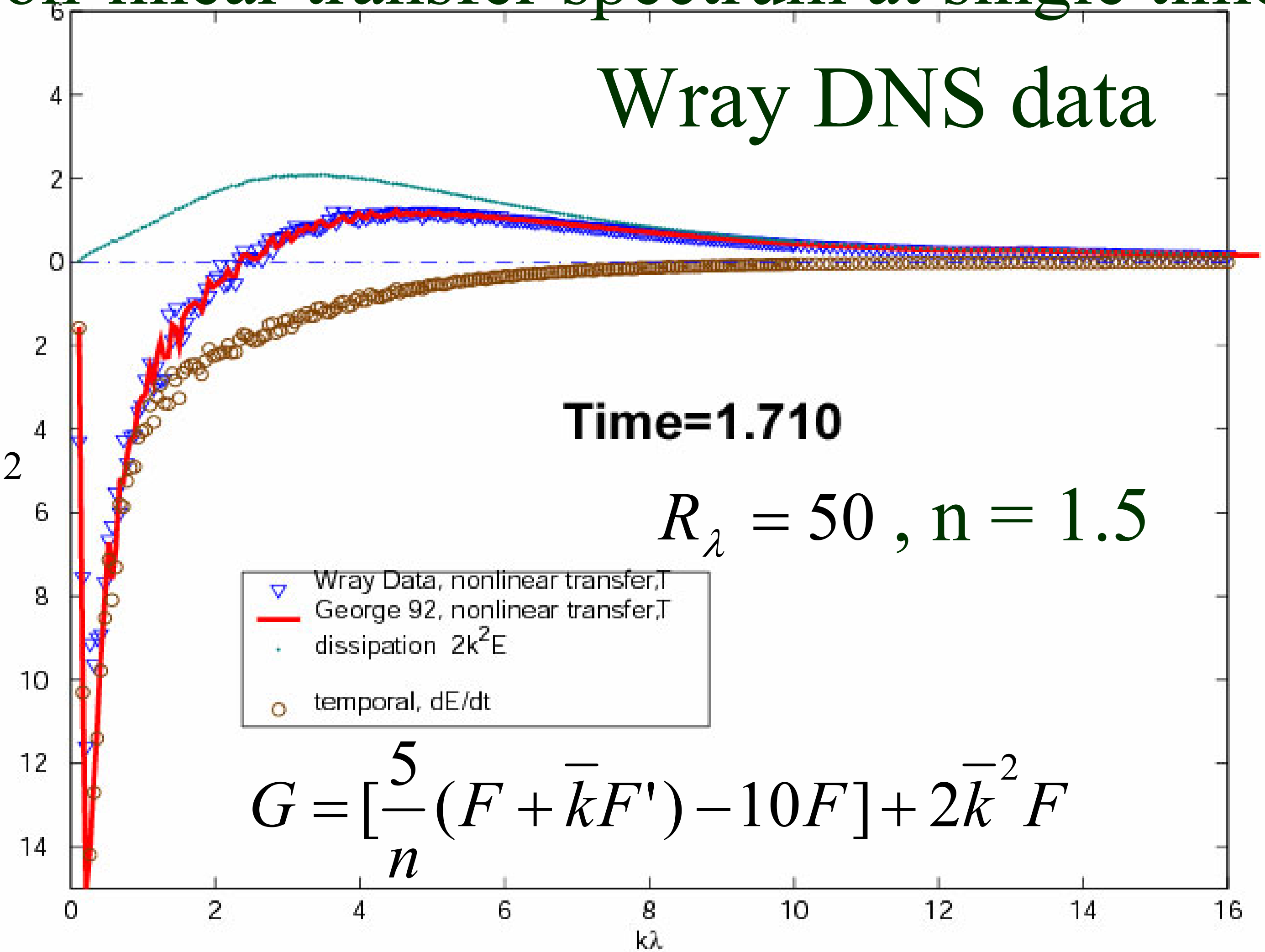
$$G = [\frac{5}{n}(F + \bar{k}F') - 10F] + 2\bar{k}^2 F$$

where $u^2 \propto t^n$ and **n** is determined by the initial conditions

• There are **no** adjustable parameters since initial conditions determine the decay exponent, **n**.

Non-linear transfer spectrum at single time

Wray DNS data



$G =$

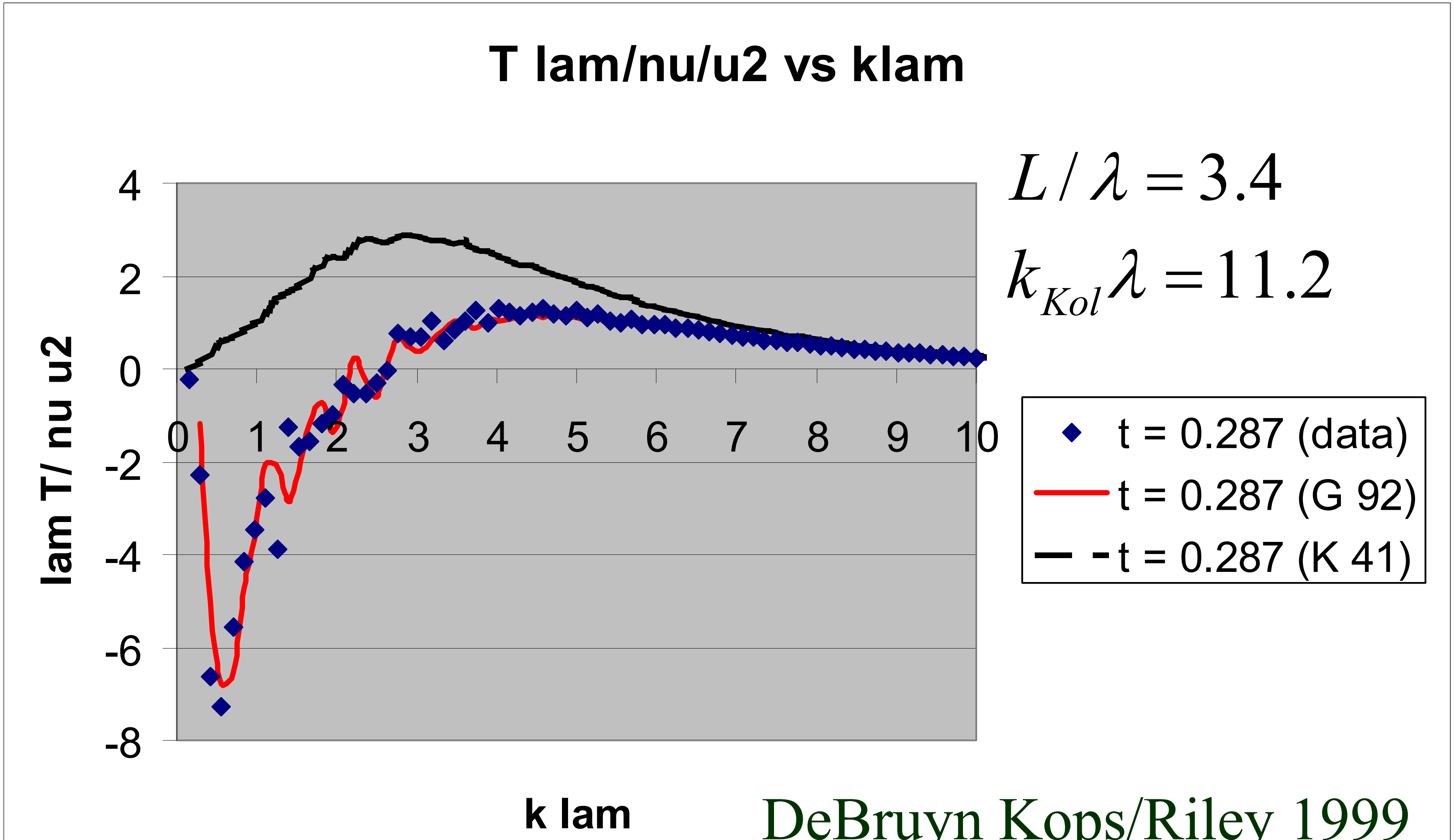
$\lambda T / \nu u^2$

$R_\lambda = 50$, n = 1.5

$$G = [\frac{5}{n}(F + \bar{k}F') - 10F] + 2\bar{k}^2 F$$

$k\lambda$

George and Wang 2002



$$L/\lambda = 3.4$$

$$k_{Kol}\lambda = 11.2$$

DeBruyn Kops/Riley 1999

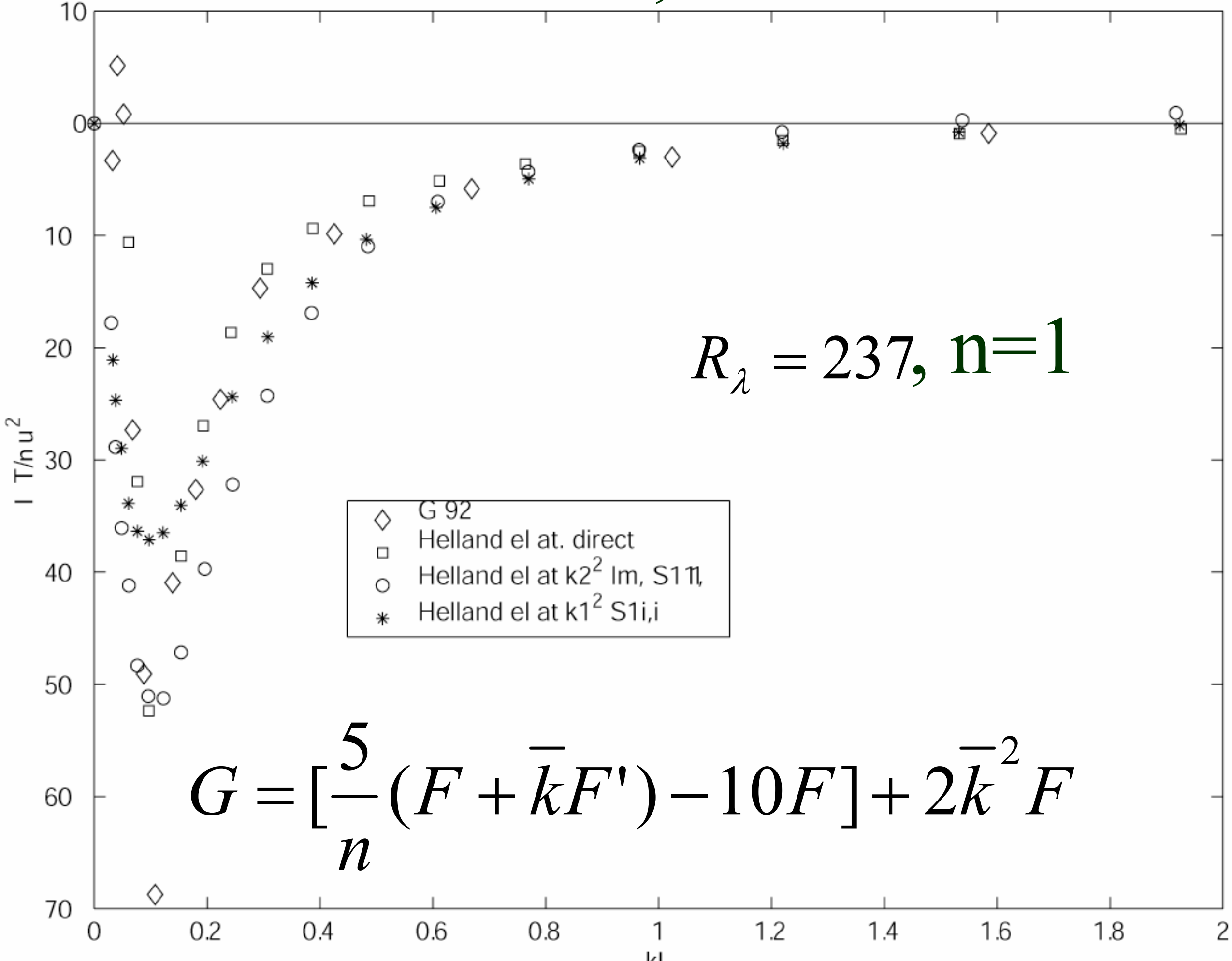
n = 1.17

George and Wang 2002

Experiments of Helland, van Atta and Stegun 1977

$G =$

$\lambda T / \nu u^2$



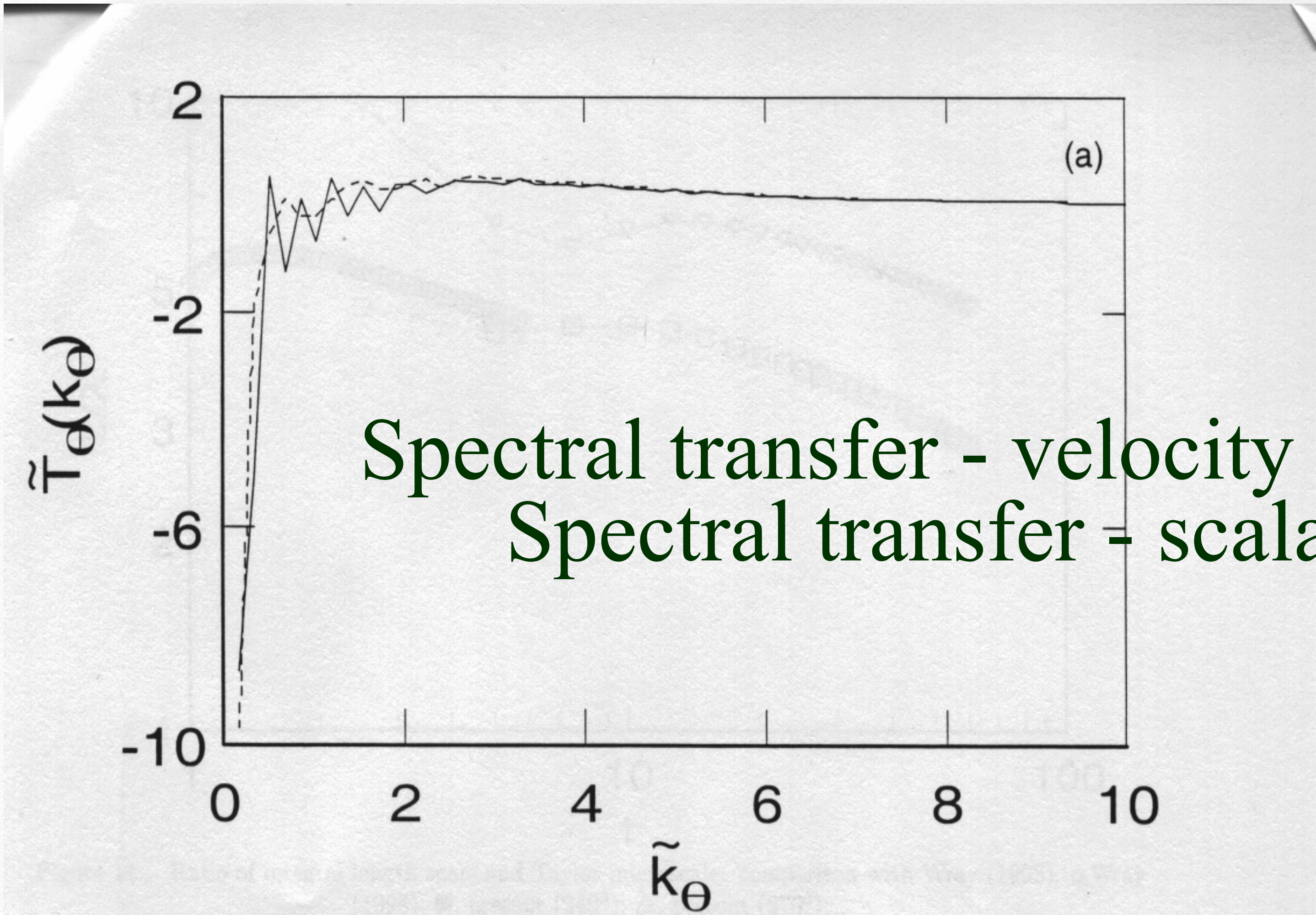
$R_\lambda = 237$, n=1

$$G = [\frac{5}{n}(F + \bar{k}F') - 10F] + 2\bar{k}^2 F$$

$k\lambda$

George and Wang 2002

DNS spectral plots from Antonia and Orlandi 2003



Spectral transfer - velocity

Spectral transfer - scalar

- In spite of this ... still disagreements about whether the derivative skewness behaves as the theory dictates.

- And there is still discussion about the behavior of the overall scaling, integral scales, etc.

- Does this reflect a problem with the simulations and experiments?

- Or is there something missing in the theory? (E.g., like there was for the axisymmetric wake.)

Implications for Turbulence Models

- Integration of equilibrium similarity spectral equations for decaying homogeneous turbulence deduces directly that: $\frac{dk}{dt} = -\varepsilon$ and $\frac{d\varepsilon}{dt} = -C_{\varepsilon_2}\frac{\varepsilon^2}{k}$

  where $C_{\varepsilon_2} = (n+1)/n$.

- **BUT** the theory also deduces that **n** is determined by the initial conditions!

Conclusions from Equilibrium Similarity

- **The k-epsilon 'model' is EXACT**, at least for homogeneous turbulence.

- **BUT** the coefficient C_{ε_2} depends on the initial conditions, in fact probably on the initial turbulence energy spectrum.

•**VERY BAD** news indeed, since there is no way to put this information into a single point turbulence model (structure-based models ??).

Summary and Conclusions

- Initial and upstream conditions affect (and even dominate) the asymptotic state of many turbulent flows.

- Equilibrium similarity theory accounts for (and expects) this behavior.

- Two-point manifestations may provide the clues and tools to understand how and why.