APS-DFD 2005

On a dynamical model for the origin of non-Gaussian statistics in turbulence

Yi Li & Charles Meneveau

Thanks to Dr. Laurent Chevillard and Prof. Greg Eyink
for many useful discussions

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Center for Environmental & Applied Fluid Mechanics

Background A, B, C, D:

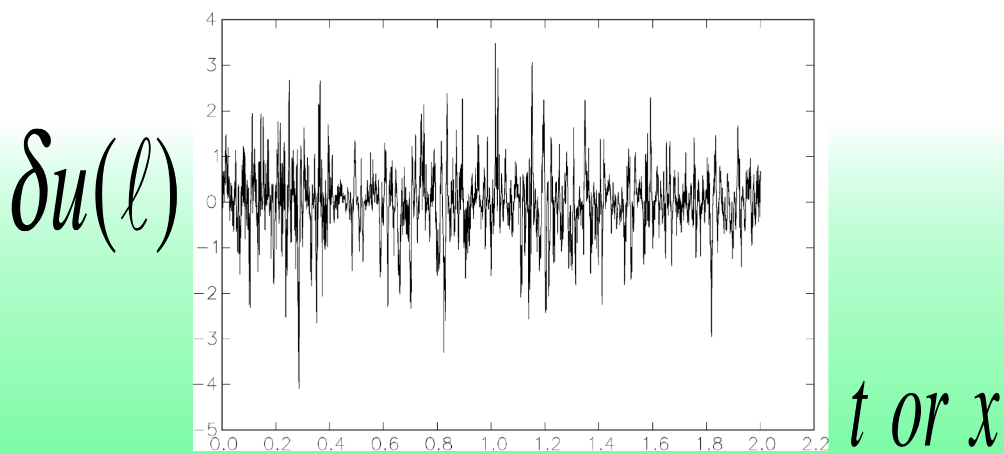
↓

Some simple math starting from N-S
+ sequence of simplifications

↓

Clearer "mechanistic" understanding of origin of
non-Gaussian statistics in turbulence

$$\delta u(\ell) = u_L(\mathbf{x} + \ell \mathbf{e}_L) - u_L(\mathbf{x})$$

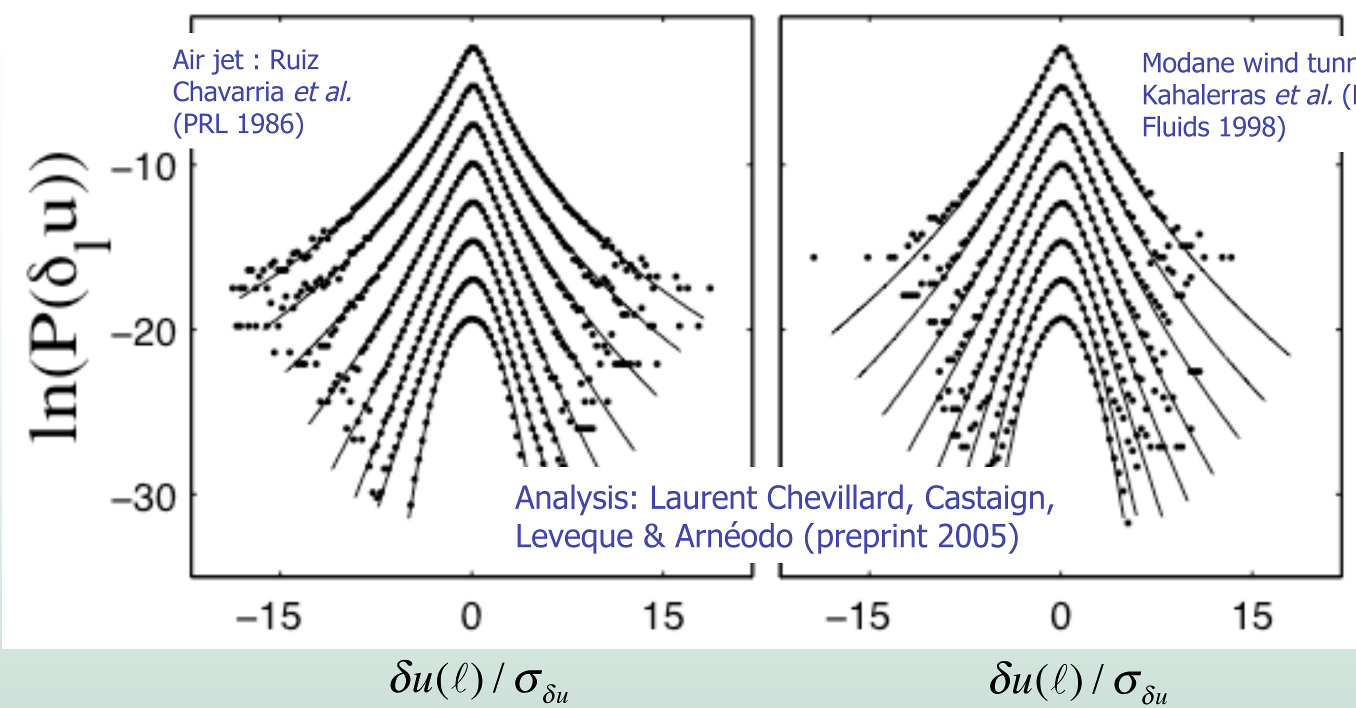


Background A:

Small-scale intermittency = **Elongated tails in PDFs** + **Anomalous scaling (clustering and non-trivial dependence on length-scale)**

Elongated tails in PDFs: (+ skewness in longitudinal direction)

$$\delta u(\ell) = u_L(\mathbf{x} + \ell \mathbf{e}_L) - u_L(\mathbf{x})$$



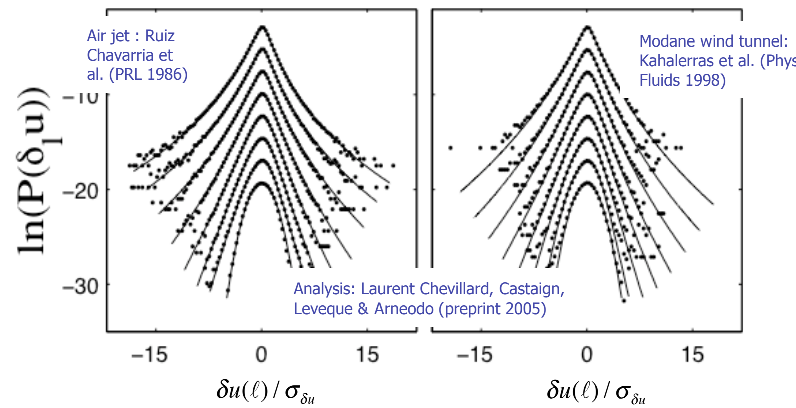
Wide body of literature, e.g.:

Anselmet *et al.*, JFM **140**, 1984
Kailasnath *et al.*, PRL **68**, 1992
Frisch, *Turbulence*, CUP 1995
Tabeling *et al.*, PRE **53**, 1995
Sreenivasan, Rev. Mod Phys. **71**, 1999
Zeff et al. Nature **421**, 2003
….
….

Background A:

Small-scale intermittency

$$\delta u(\ell) = u_L(\mathbf{x} + \ell \mathbf{e}_L) - u_L(\mathbf{x})$$



Observations:

- Non-Gaussian tails are very robust
- They occur in DNS, even at small Re turbulence
- They appear in ad-hoc turbulence models
  (not systematically derived from N-S):
  - Shell models of turbulence (see Biferale Annu. Rev. Fluid Mech. 2003)
  - Mapping closure (Kraichnan PRL **65**, 1990; She & Orszag PRL **66**, 1991)
- Simple mechanistic explanation “elusive” in 3D
- Do not occur in 2D turbulence (but see 1-D Burgers equation…)

Background B:

1-D Burgers equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

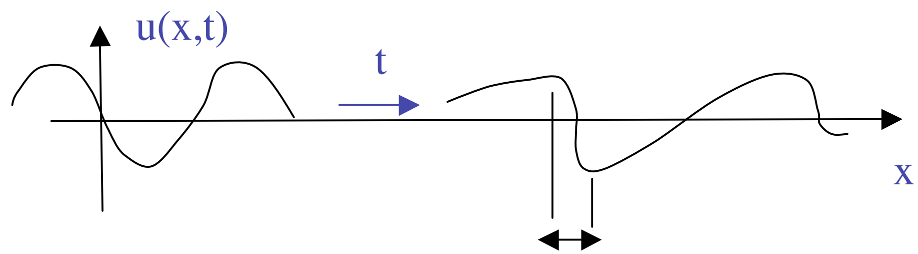
$$A = \frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = \frac{\partial A}{\partial t} + u\frac{\partial A}{\partial x} + AA$$

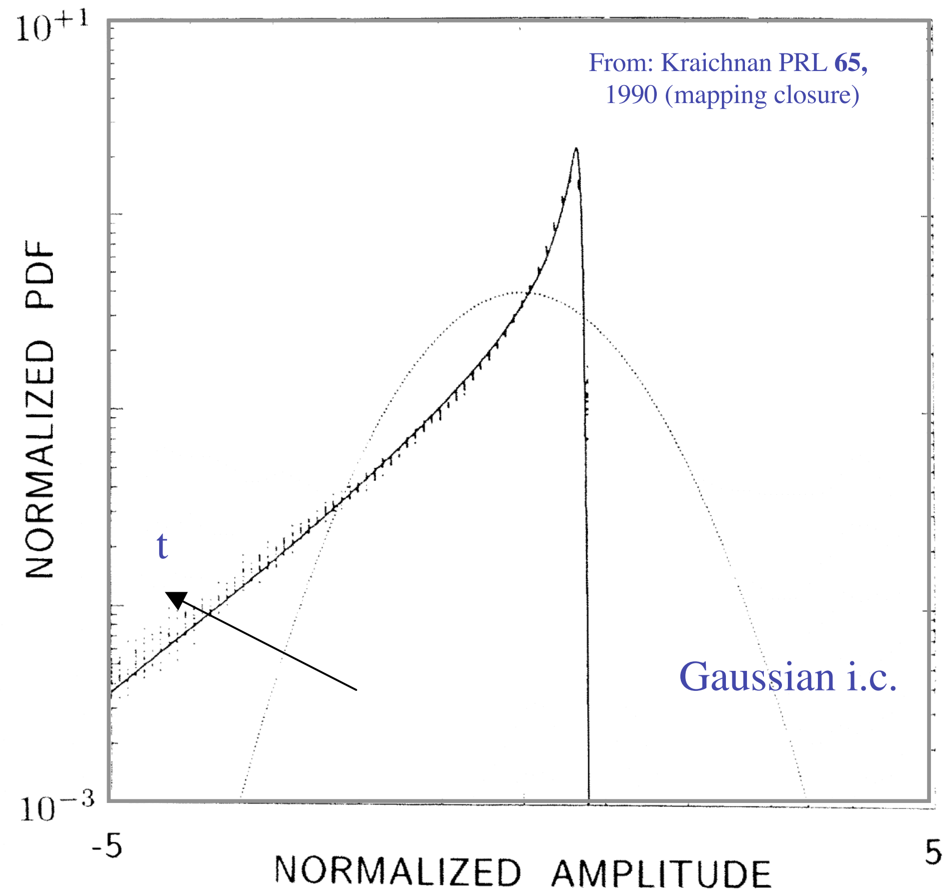
$$\frac{dA}{dt} = -A^2$$

$$\delta u(\ell) \equiv A\ell$$

$$\frac{d\delta u}{dt} = -\delta u^2 \ell^{-1}$$



Intense negative gradient occurs over smaller fraction of domain



Here is a "trivial" picture of origin of non-Gaussian tails and skewness: free particle motion. What is the 3-D analogue of this? Problems: which direction? Multiple velocity components..

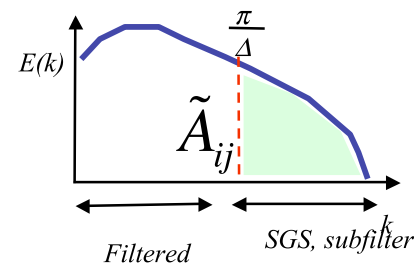
Background C:

Restricted Euler dynamics in (inertial range of) turbulence:

Restricted Euler: Vieillefosse, Phys. A, **125**, 1985
Cantwell, Phys. Fluids A**4**, 1992
Filtered turbulence: Borue & Orszag, JFM **366**, 1998
Van der Bos *et al.*, Phys Fluids **14**, 2002:

- Filtered Navier-Stokes equations:

$$\frac{\partial \tilde{u}_j}{\partial t} + \tilde{u}_k \frac{\partial \tilde{u}_j}{\partial x_k} = -\frac{\partial \tilde{p}}{\partial x_j} + \nu \nabla^2 \tilde{u}_j - \frac{\partial}{\partial x_k} \tau_{jk}$$



- Take gradient: $\tilde{A}_{ij} = \frac{\partial \tilde{u}_j}{\partial x_i}$

$$\frac{\partial \tilde{A}_{ij}}{\partial t} + \tilde{u}_k \frac{\partial \tilde{A}_{ij}}{\partial x_k} \equiv$$

$$\frac{d\tilde{A}_{ij}}{dt} = -\left(\tilde{A}_{ik}\tilde{A}_{kj} - \frac{\delta_{ij}}{3}\tilde{A}_{mk}\tilde{A}_{km}\right) + \left(-\frac{\partial^2 p}{\partial x_i \partial x_j} + \tfrac{1}{3}\frac{\partial^2 p}{\partial x_k \partial x_k}\delta_{ij}\right) - \left(\frac{\partial^2 \tau^d_{kj}}{\partial x_i \partial x_k} - \frac{\delta_{ij}}{3}\frac{\partial^2 \tau^d_{kl}}{\partial x_l \partial x_k}\right) + \nu \nabla^2 \tilde{A}_{ij}$$

Self-interaction

Pressure Hessian

Subgrid-scale
(+ viscous) effects

$$\frac{d\tilde{A}_{ij}}{dt} = -\left(\tilde{A}_{ik}\tilde{A}_{kj} - \frac{\delta_{ij}}{3}\tilde{A}_{mk}\tilde{A}_{km}\right) + H_{ij}$$

Background C:

Restricted Euler dynamics $H_{ij} = 0$ in (inertial range of) turbulence:

- **Invariants (Cantwell 1992):**

$$Q_\Delta \equiv -\frac{1}{2}\tilde{A}_{ki}\tilde{A}_{ik}$$

$$R_\Delta \equiv -\frac{1}{3}\tilde{A}_{km}\tilde{A}_{mn}\tilde{A}_{nk}$$

$$\tilde{A}_{ji}\frac{d\tilde{A}_{ij}}{dt} = \tilde{A}_{ji}(\tilde{A}_{ik}\tilde{A}_{kj} - \tfrac{1}{3}\tilde{A}_{mk}\tilde{A}_{km}\delta_{ij}) \rightarrow \frac{dQ_\Delta}{dt} = -3R_\Delta$$

$$\tilde{A}_{jk}\tilde{A}_{ki}\frac{d\tilde{A}_{ij}}{dt} = \tilde{A}_{jk}\underbrace{\tilde{A}_{ki}(\tilde{A}_{ik}\tilde{A}_{kj}}_{} - \tfrac{1}{3}\tilde{A}_{mk}\tilde{A}_{km}\delta_{ij}) \rightarrow \frac{dR_\Delta}{dt} = \frac{2}{3}Q_\Delta^2$$

Cayley-Hamilton Theorem

$$A_{ik}A_{kn}A_{nj} + PA_{ik}A_{kj} + QA_{ij} + R\delta_{ij} = 0$$

Remarkable projection (decoupling)!

More literature:

Equations for all 5 invariants:

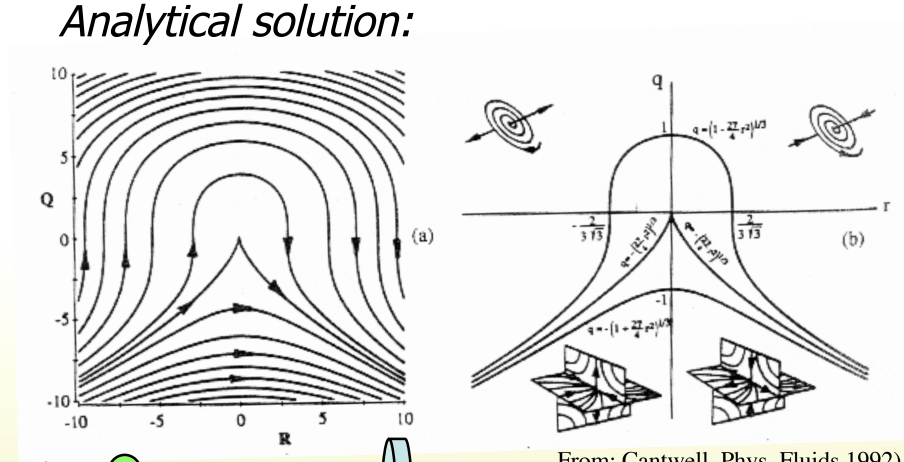
Martin, Dopazo & Valiño (Phys. Fluids, 1998)

Equations for eigenvalues,

and higher-dimensional versions:

Liu & Tadmor (Commun. Math. Phys., 2002)

Analytical solution:



From: Cantwell, Phys. Fluids 1992)

- **Singularity in finite time,** but

  - Predicts preference for axisymmetric expansion

  - Predicts alignment of vorticity with intermediate eigenvector of S: β_s

Background D:

Models for pressure-viscous-SGS Hessian:

- Stochastic differential equation (H_{ij} constructed such that $\tilde{A}_{ij}\tilde{A}_{ij}$ is lognormal with imposed variance, yields stationary system, Girimaji & Pope, Phys. Fluids A**2**, 1990)

- Model H_{ij} by keeping track of material deformations:

  - Tetrad dynamics (Chertkov, Pumir & Shraiman, Phys. Fluids **11**, 1999; Nasso et al..)

  - Cauchy-Green tensor evolution (Jeong & Girimaji, Theor. Comp. Fluid Dyn. **16**, 2003)

- At the cost of solving N > 8 ODEs (stochastic or deterministic), all these models predict intermittency and skewness (plus other things, such as vorticity alignment trends,…)

- But "large" N precludes "text-book simple" insight into formation of non-Gaussian tails…

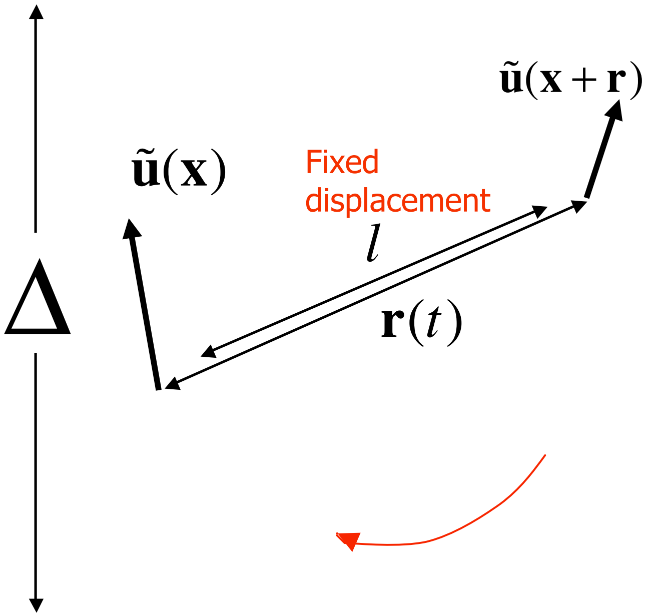
We seek particular “projections” of Restricted Euler dynamics that could illuminate formation of non-Gaussian tails,

i.e. “are there any other simple ODE's like

$$\begin{cases} \dfrac{dQ_\Delta}{dt} = -3R_\Delta \\ \dfrac{dR_\Delta}{dt} = \dfrac{2}{3}Q_\Delta^2 \end{cases}$$

??”

Velocity increments: Lagrangian evolution



$$\tilde{u}_i(\mathbf{x}+\mathbf{r}) - \tilde{u}_i(\mathbf{x}) = \tilde{A}_{ki} r_k + O(r^2)$$

Longitudinal

$$\delta u \equiv \tilde{A}_{ki} r_k \frac{r_i}{r} \frac{\ell}{r}$$

$$\delta u(t) \equiv \tilde{\mathbf{A}}(t) : (\hat{\mathbf{r}}(t)\hat{\mathbf{r}}(t))\ \ell = \tilde{A}_{rr}\ell$$

Transverse

$$\delta v^2 \equiv \left[\left(\delta_{ij} - \frac{r_i r_j}{r^2} \right) \tilde{A}_{kj} r_k \frac{\ell}{r} \right]^2$$

See: Yi & Meneveau, Phys. Rev. Lett. **95**, 164502, Oct. 2005

Velocity increments: Lagrangian evolution

$$\delta u \equiv \tilde{A}_{ki} r_k \frac{r_i}{r}\frac{\ell}{r} \qquad\qquad \delta v^2 \equiv \left[\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\tilde{A}_{kj} r_k \frac{\ell}{r}\right]^2$$

Rate of change of longitudinal velocity increment, following the flow (both end-points in linearized flow):

$$\frac{d}{dt}\delta u = \frac{d}{dt}\left(\tilde{A}_{ki} r_k \frac{r_i}{r}\frac{\ell}{r}\right) = \frac{d\tilde{A}_{ki}}{dt}\frac{r_k r_i}{r}\frac{\ell}{r} + \tilde{A}_{ki}\frac{dr_k}{dt}\frac{r_i}{r}\frac{\ell}{r} + \tilde{A}_{ki}\frac{dr_i}{dt}\frac{r_k}{r}\frac{\ell}{r} - 2\tilde{A}_{ki}\frac{r_k r_i}{r^3}\frac{dr}{dt}\ell$$

$$\frac{d\tilde{A}_{ij}}{dt} = -(\tilde{A}_{ik}\tilde{A}_{kj} - \tfrac{1}{3}\tilde{A}_{mk}\tilde{A}_{km}\delta_{ij}) + H_{ij}$$

$$\frac{dr_i}{dt} = \frac{\partial \tilde{u}_i}{\partial x_m} r_m = \tilde{A}_{mi} r_m$$

$$\frac{d}{dt}\delta u = -(\tilde{A}_{km}\tilde{A}_{mi} - \tfrac{1}{3}\tilde{A}_{pq}\tilde{A}_{qp}\delta_{ki})\frac{r_k r_i}{r}\frac{\ell}{r} + \tilde{A}_{ki}\tilde{A}_{mk} r_m \frac{r_i}{r}\frac{\ell}{r} + \tilde{A}_{ki}\tilde{A}_{mi} r_m \frac{r_k}{r}\frac{\ell}{r} - 2\tilde{A}_{ki}\frac{r_k r_i}{r^3}\frac{r_m}{r}\ell\tilde{A}_{pm} r_p + H_{mn}\frac{r_m r_n}{r}\frac{\ell}{r}$$

$$\frac{d}{dt}\delta u = \left[\left(\delta_{ij} - \frac{r_i r_j}{r^2}\right)\tilde{A}_{kj} r_k \frac{\ell}{r}\right]^2 \frac{1}{\ell} - \left(\tilde{A}_{ki} r_k \frac{r_i}{r}\frac{\ell}{r}\right)^2 \frac{1}{\ell} + \tfrac{1}{3}\tilde{A}_{pq}\tilde{A}_{qp}\ell + H_{mn}\frac{r_m r_n}{r^2}\ell$$

$$\frac{d}{dt}\delta u = \frac{1}{\ell}\left(\delta v^2 - \delta u^2\right) - \frac{2}{3}Q\ell + H_{mn}\frac{r_m r_i}{r^2}\ell \qquad\qquad H_{mn} = \left(\frac{\partial^2}{\partial x_m \partial x_k}\left[p\delta_{kn} - \tau_{kn}^{SGS} + 2\nu\tilde{S}_{kn}\right]\right)^{anisotropic}$$

Velocity increments: Lagrangian evolution

$$\frac{d}{dt}\delta u = \frac{1}{\ell}\left(\delta v^2 - \delta u^2\right) - \frac{2}{3}Q\ell + H_{mn}\frac{r_m r_i}{r^2}\ell$$

From a similar derivation for δv:

$$\frac{d}{dt}\delta v = -\frac{1}{\ell}\delta u \delta v + H_{mn} r_m e_n ,$$

$$H_{mn} = \left(\frac{\partial^2}{\partial x_m \partial x_k}\left[p\delta_{kn} - \tau_{kn}^{SGS} + 2\nu \tilde{S}_{kn} \right] \right)^{anisotropic}$$

$$e_n = \frac{\delta u_i \left(\delta_{in} - \frac{r_i r_n}{r^2} \right)}{\delta v}$$

Restricted Euler simplification: $H_{ij} = 0$

Velocity increments under Restricted Euler, at **fixed displacement length** (linear vel. field):

4 ODEs

$$\begin{cases} \dfrac{d}{dt}\delta u = \dfrac{1}{\ell}\left(\delta v^2 - \delta u^2\right) - \dfrac{2}{3}Q\ell \\[2ex] \dfrac{d}{dt}\delta v = -\dfrac{2}{\ell}\ \delta u\ \delta v \\[2ex] \dfrac{d}{dt}Q = -3R \\[2ex] \dfrac{d}{dt}R = \dfrac{2}{3}Q^2 \end{cases}$$

$$H_{ij} = 0$$

Numerical experimentation shows
most basic trends towards intermittency
not very sensitive to Q-coupling

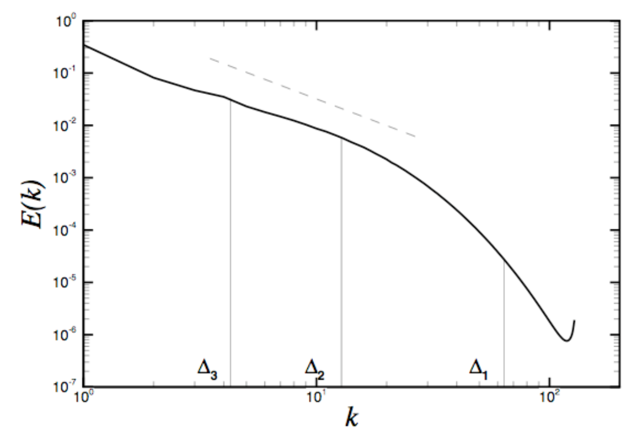
Advected delta-vee equations:

$$\begin{cases} \dfrac{d}{dt}\delta u = \dfrac{1}{\ell}\left(\delta v^2 - \delta u^2\right) - \dfrac{2}{3}Q_0\ell \\ \dfrac{d}{dt}\delta v = -\dfrac{2}{\ell}\ \delta u\ \delta v \end{cases}$$

See: Yi & M, Phys. Rev. Lett. **95**, 164502, Oct. 2005

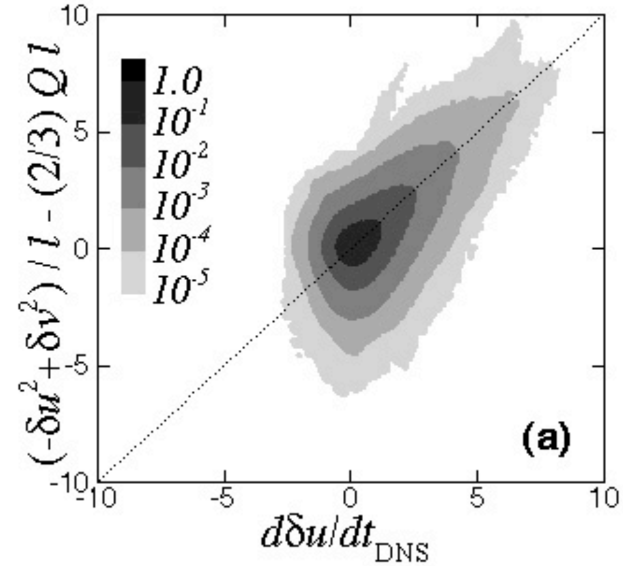
Does this simple system have ANY resemblance to what happens in turbulence?

256^3 DNS, filtered at 40η, evaluated Lagrangian rate of change of velocity increments numerically, and compared to RHS of advected delta-vee equations

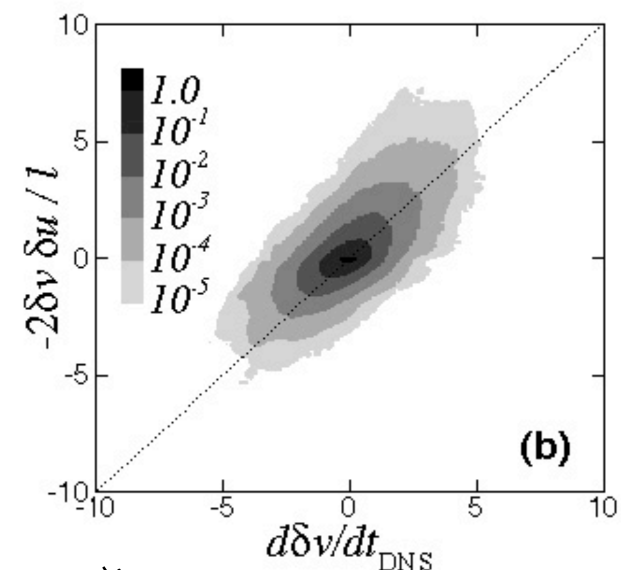


Comparison with DNS, Lagrangian rate of change of velocity increments:

$$\begin{cases} \dfrac{d}{dt}\delta u = \dfrac{1}{\ell}\left(\delta v^2 - \delta u^2\right) - \dfrac{2}{3}Q\ell \\ \dfrac{d}{dt}\delta v = -\dfrac{2}{\ell}\ \delta u\ \delta v \end{cases}$$



$\rho = 0.54$



$\rho = 0.61$

256^3 DNS, filtered at 40η, Δ=40 η, evaluated δu, δv, and their Lagrangian rate of change of velocity increments numerically

Evolution from Gaussian initial conditions, $Q_0=0$:

Initial condition:

δu = Gaussian zero mean, unit variance

δv_k = Gaussian zero mean, unit variance, k=1,2

$$\delta v = \sqrt{\delta v_1^2 + \delta v_2^2}$$

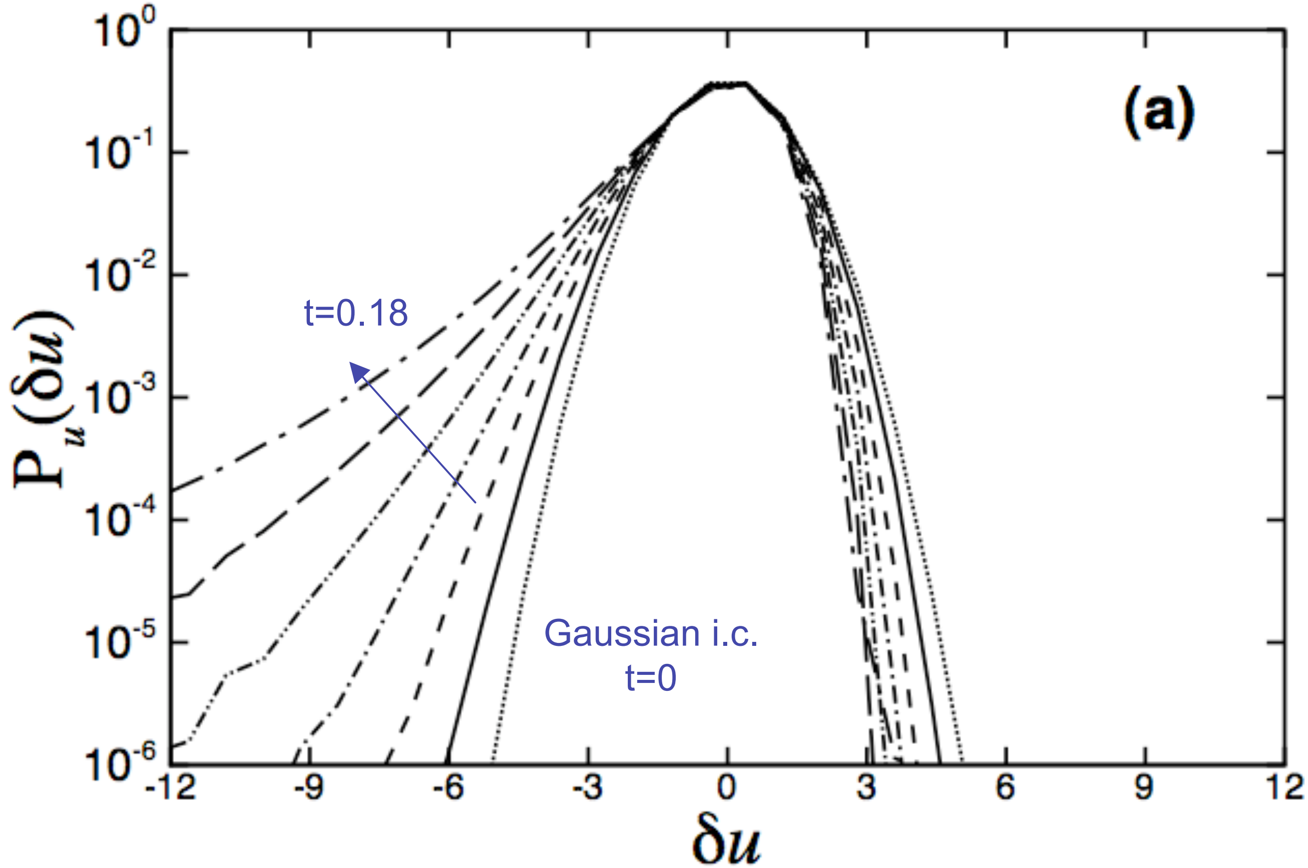
set $\ell = 1$ and $Q_0 = 0$

Non-dimensional:

$$\begin{cases} \dfrac{d}{dt}\delta u = \delta v^2 - \delta u^2 \\ \dfrac{d}{dt}\delta v = -2\delta u \ \delta v \end{cases}$$

Evolution from Gaussian initial conditions, $Q_0=0$:

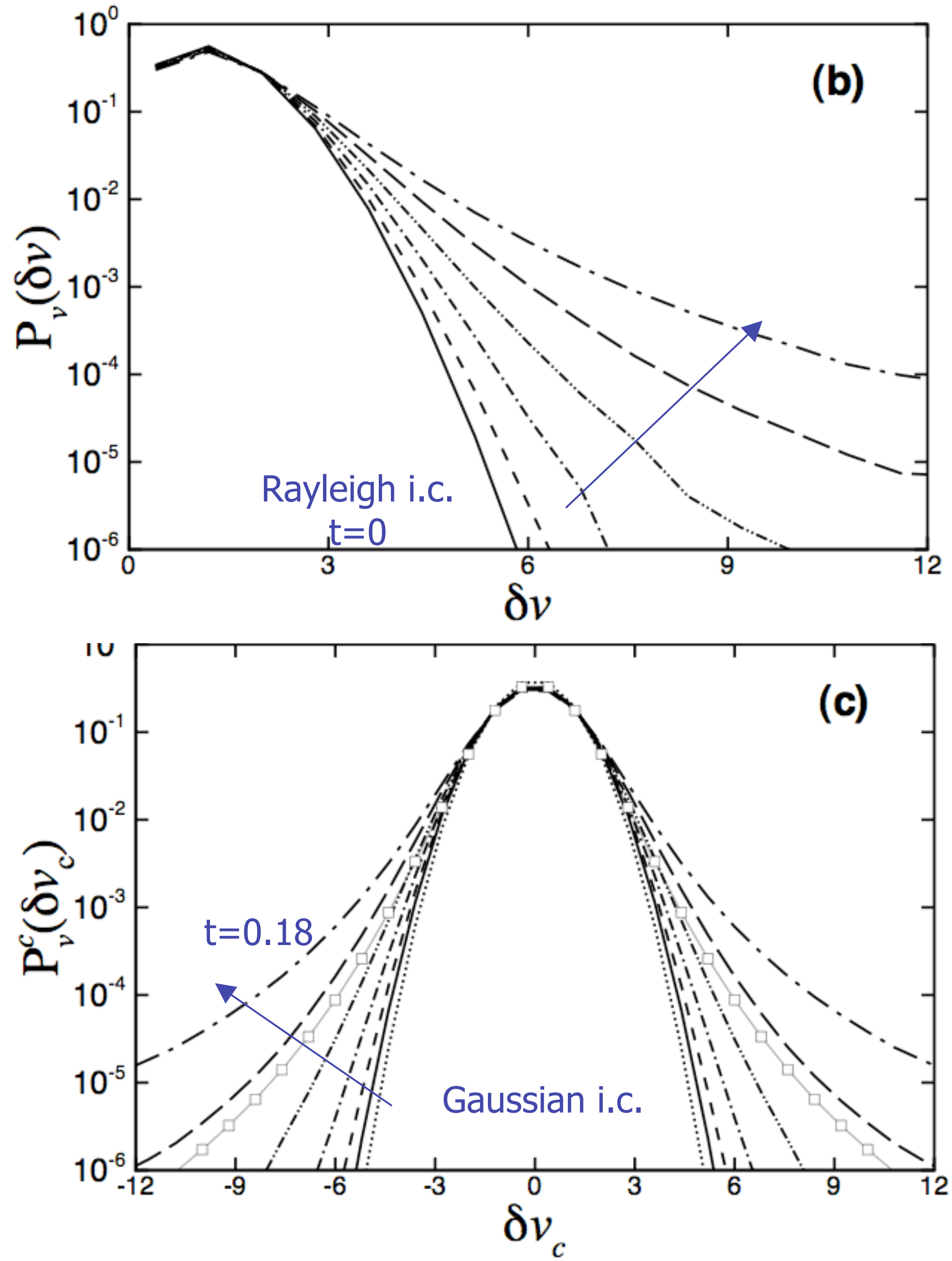
$$\begin{cases} \frac{d}{dt}\delta u = \delta v^2 - \delta u^2 \\ \frac{d}{dt}\delta v = -2\delta u\ \delta v \end{cases}$$



Evolution from Gaussian initial conditions, $Q_0=0$:

$$
\begin{cases}
\dfrac{d}{dt}\delta u = \delta v^2 - \delta u^2 \\
\dfrac{d}{dt}\delta v = -2\delta u \ \delta v
\end{cases}
$$

$$|\delta v|$$



Individual component:
θ is random,
uniformly distributed in [0, 2p)

$$\delta v_c = |\delta v| \cos\theta$$

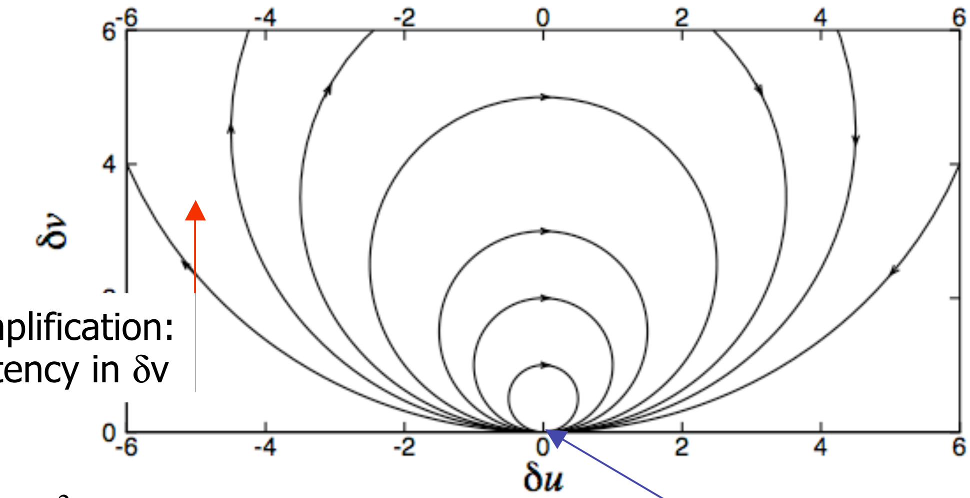
$$P_v^c(\delta v_c) = \frac{1}{\pi} \int_{|\delta v_c|}^{\infty} \frac{P_v(\delta v)}{\sqrt{\delta v^2 - \delta v_c^2}} d\delta v$$

Basic properties: Phase-space & invariant

$$\begin{cases} \frac{d}{dt}\delta u = \delta v^2 - \delta u^2 \\ \frac{d}{dt}\delta v = -2\delta u \ \delta v \end{cases}$$

Bruno Eckhhardt & David Levermore's observation:

$$z = \delta u + i\delta v \rightarrow \frac{dz}{dt} = -z^2$$



Cross-amplification: Intermittency in δv

Self-amplification: Skewness in δu

"Degenerate saddle-node of index 2 (Guckenheimer & Holmes)"

Invariant of advected delta-vee system:

$$\frac{d\delta u}{d\delta v} = \frac{1}{2}\frac{\delta u}{\delta v} - \frac{1}{2}\frac{\delta v}{\delta u}$$

$$\frac{dK}{d\delta v} = \frac{K}{\delta v} - \frac{1}{2}\delta v$$

$$K = \frac{1}{2}\delta u^2$$

$$K = \delta v F(\delta v)$$

$$\delta v \frac{dF}{d\delta v} + F = F - \frac{1}{2}\delta v$$

$$F = -\frac{1}{2}\delta v + U$$

$$U = \frac{1}{2}\delta u^2 / \delta v + \frac{1}{2}\delta v = \boxed{\frac{1}{2}\left(\delta u^2 + \delta v^2\right)\frac{1}{\delta v}}$$

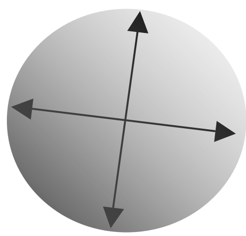
$$U = \frac{1}{2}\left(\delta u^2 + \delta v^2\right)\frac{1}{\delta v}$$

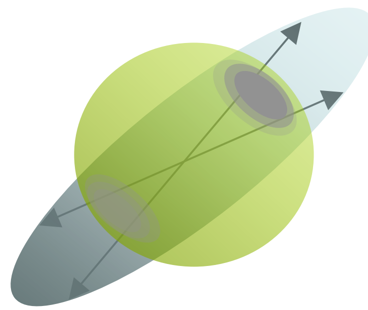
"For small initial δv (particles moving directly towards each other), gradient can become arbitrarily large at later times"

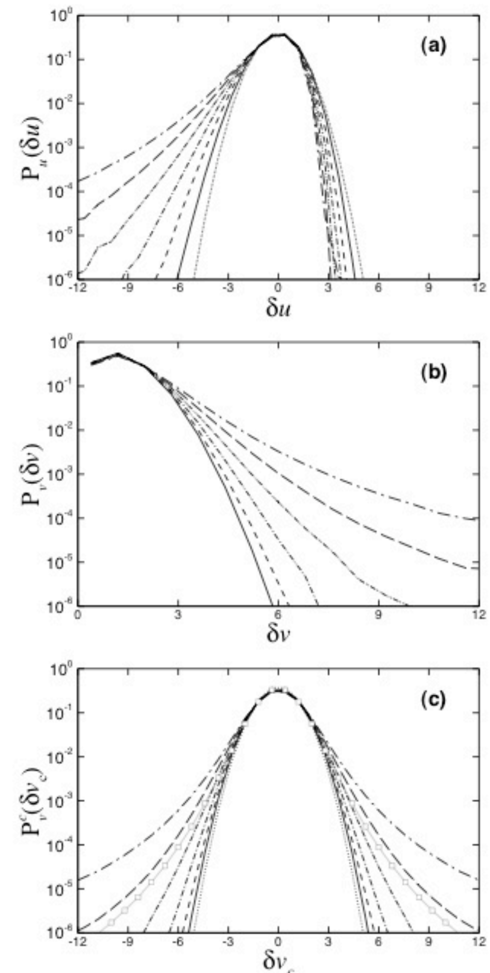
Alignment bias correction factor:

(thanks to Greg Eyink for pointing out the need for a correction)

$$\mathbf{r}(0), \quad |\mathbf{r}(0)| = \ell \qquad\qquad \mathbf{r}(t)$$







$$\ell^3 d\Omega_0 = r(t)^3 d\Omega(t) \qquad \frac{d\Omega(t)}{d\Omega_0} = \left(\frac{\ell}{r(t)}\right)^3$$

$$\frac{dr}{dt} = \delta u(r,t) = \delta u \cdot \left(\frac{r}{\ell}\right)$$

$$\frac{d}{dt}\ln(r/\ell) = \delta u/\ell$$

$$\frac{d\Omega(t)}{d\Omega_0} = \exp\left(-3\ell^{-1}\int_0^t \delta u(t')dt'\right) \qquad P \rightarrow P\frac{d\Omega(t)}{d\Omega_0}$$

Can be evaluated from advected delta-vee system

See: Yi & Meneveau,
Phys. Rev. Lett. **95**, 164502,

Effects of neglected terms:

$$\begin{cases} \dfrac{d}{dt}\delta u = \dfrac{1}{\ell}\left(\delta v^2 - \delta u^2\right) - \dfrac{2}{3}Q\ell + H_{mn}\dfrac{r_m r_i}{r^2}\ell \\ \dfrac{d}{dt}\delta v = -\dfrac{1}{\ell}\delta u \delta v \quad + \quad H_{mn}\dfrac{r_m e_n}{r}\ell, \end{cases}$$

$$H_{mn} = \left(\frac{\partial^2}{\partial x_m \partial x_k}\left[p\delta_{kn} - \tau_{kn}^{SGS} + 2\nu\tilde{S}_{kn}\right]\right)^{anisotropic}$$

Effects of *Q*-term: continuity

$$Q = -\frac{1}{2}\tilde{A}_{ij}\tilde{A}_{ji}$$

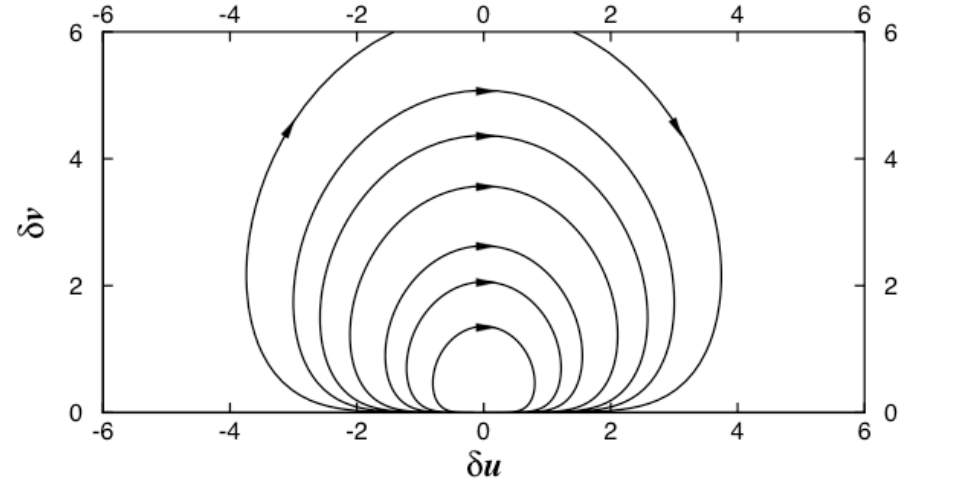
$$Q = -\frac{1}{2}\left(\tilde{A}_{rr}\tilde{A}_{rr} + \tilde{A}_{\theta\theta}\tilde{A}_{\theta\theta} + \left(-\tilde{A}_{rr} - \tilde{A}_{\theta\theta}\right)^2 + ...\right)$$

$$Q = -\frac{1}{2}\left(2\underbrace{\tilde{A}_{rr}\tilde{A}_{rr}} + \tilde{A}_{\theta\theta}\tilde{A}_{\theta\theta} + ...\right)$$

$$\rightarrow -\frac{2}{3}Q = +\frac{2}{3}\delta u^2 \frac{1}{\ell^2} - \frac{2}{3}Q^*$$

$$Q^* = 0, \quad H_{ij} = 0$$

$$\begin{cases} \dfrac{d}{dt}\delta u = \dfrac{1}{\ell}\left(\delta v^2 - \boxed{\dfrac{1}{3}}\delta u^2\right) \\ \dfrac{d}{dt}\delta v = -\dfrac{1}{\ell}\delta u \delta v \end{cases}$$



Effects of neglected terms:

$$
\begin{cases}
\dfrac{d}{dt}\delta u = \dfrac{1}{\ell}\left(\delta v^2 - \delta u^2\right) - \dfrac{2}{3}Q\ell + H_{mn}\dfrac{r_m r_i}{r^2}\ell \\[2ex]
\dfrac{d}{dt}\delta v = -\dfrac{1}{\ell}\delta u \delta v \quad + \quad H_{mn}\dfrac{r_m e_n}{r}\ell,
\end{cases}
$$

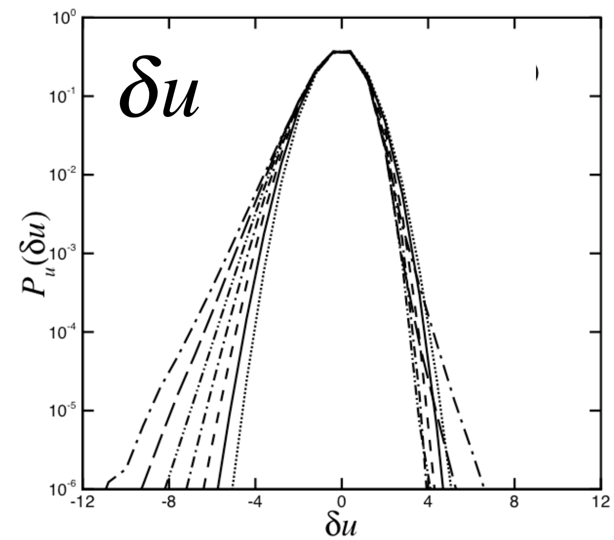
Effects of Q-term: continuity

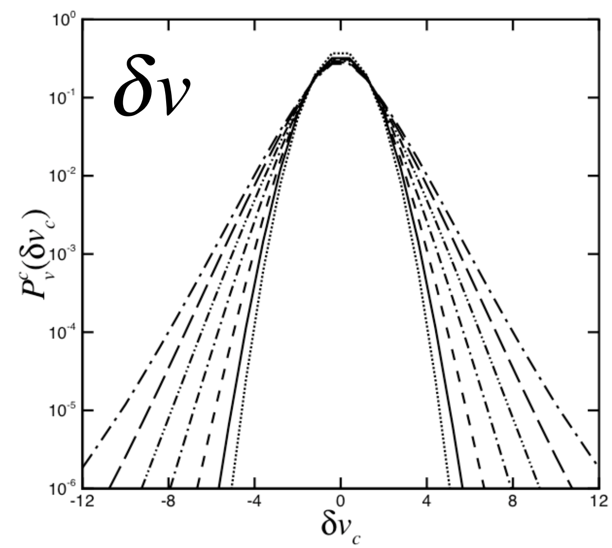
$$Q = -\frac{1}{2}\left(\tilde{A}_{rr}\tilde{A}_{rr} + \tilde{A}_{\theta\theta}\tilde{A}_{\theta\theta} + \left(-\tilde{A}_{rr} - \tilde{A}_{\theta\theta}\right)^2 + \ldots.\right)$$

$$Q = -\frac{1}{2}\left(2\underbrace{\tilde{A}_{rr}\tilde{A}_{rr}} + \tilde{A}_{\theta\theta}\tilde{A}_{\theta\theta} + \ldots\right)$$

$$\rightarrow -\frac{2}{3}Q = +\frac{2}{3}\delta u^2 \frac{1}{\ell^2} - \frac{2}{3}Q^*$$

$$
\begin{cases}
\dfrac{d}{dt}\delta u = \dfrac{1}{\ell}\left(\delta v^2 - \boxed{\dfrac{1}{3}}\delta u^2\right) \\[2ex]
\dfrac{d}{dt}\delta v = -\dfrac{1}{\ell}\delta u \delta v
\end{cases}
$$



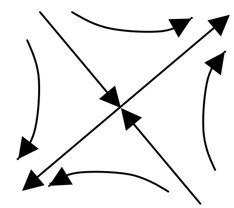


Effects of dimensionality on *Q*-term:

$$\begin{cases} \dfrac{d}{dt}\delta u = \dfrac{1}{\ell}\left(\delta v^2 - \delta u^2\right) - \dfrac{2}{3}Q\ell + H_{mn}\dfrac{r_m r_i}{r^2}\ell \\ \dfrac{d}{dt}\delta v = -\dfrac{1}{\ell}\delta u \delta v \quad + \quad H_{mn}\dfrac{r_m e_n}{r}\ell, \end{cases}$$

$$H_{mn} = \left(\frac{\partial^2}{\partial x_m \partial x_k}\left[p\delta_{kn} - \tau_{kn}^{SGS} + 2\nu\tilde{S}_{kn}\right]\right)^{anisotropic}$$

D=2:



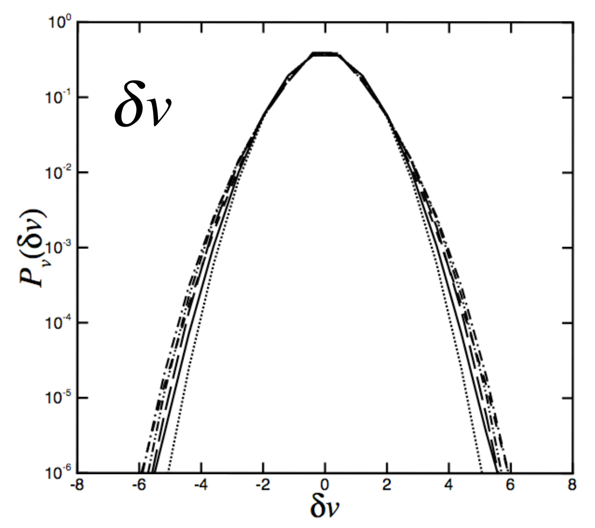
$$\frac{d}{dt}\delta u = \frac{1}{\ell}\left(\delta v^2 - \delta u^2\right) - \det(\tilde{\mathbf{A}})\ell + \ldots$$

$$\det(\tilde{\mathbf{A}}) = \tilde{A}_{rr}(-\tilde{A}_{rr}) + \ldots.$$

$$-\det(\tilde{\mathbf{A}})\ell \rightarrow +\delta u^2 \frac{1}{\ell}$$

Very strong directional coupling due to pressure
No intermittency in δ-velocities

Cancels the self-amplification of negative δu



Restricted Euler in 2D: no self-stretching of gradient (i.e. Gaussian I.C. remains Gaussian…)

$$\frac{dA_{ij}}{dt} = 0$$

Effects of dimensionality on *Q*-term:

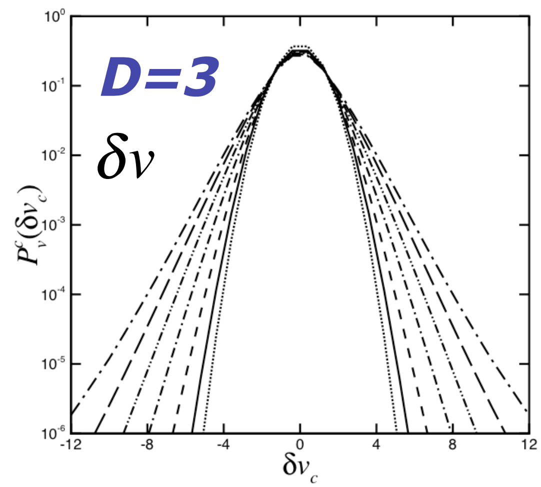
$$\begin{cases} \frac{d}{dt}\delta u = \frac{1}{\ell}\left(\delta v^2 - \delta u^2\right) - \frac{2}{3}Q\ell + H_{mn}\frac{r_m r_i}{r^2}\ell \\ \frac{d}{dt}\delta v = -\frac{1}{\ell}\delta u \delta v \quad + \quad H_{mn}\frac{r_m e_n}{r}\ell, \end{cases}$$

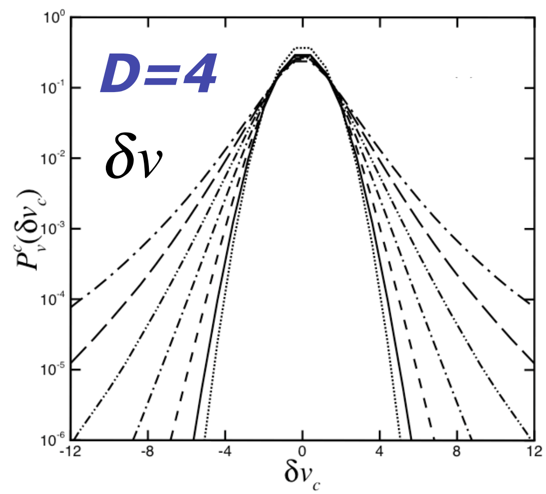
$$H_{mn} = \left(\frac{\partial^2}{\partial x_m \partial x_k}\left[p\delta_{kn} - \tau_{kn}^{SGS} + 2\nu \tilde{S}_{kn}\right]\right)^{anisotropic}$$

For general *D*:

$$\frac{d}{dt}\delta u = \frac{1}{\ell}\left[\delta v^2 - \left(1 - \frac{2}{D}\right)\delta u^2\right] - \ldots Q^* + \ldots$$

The higher *D*, the weaker the coupling with other directions, more tendency towards long tails from advected delta-vee system





Suzuki et al. (Phys. Fluids **17**, 2005):
DNS of 64^4 and 128^4 turbulence shows PDFs of δu in 4-D a bit more intermittent than in 3-D

Effects of pressure Hessian, SGS and viscous force gradients :

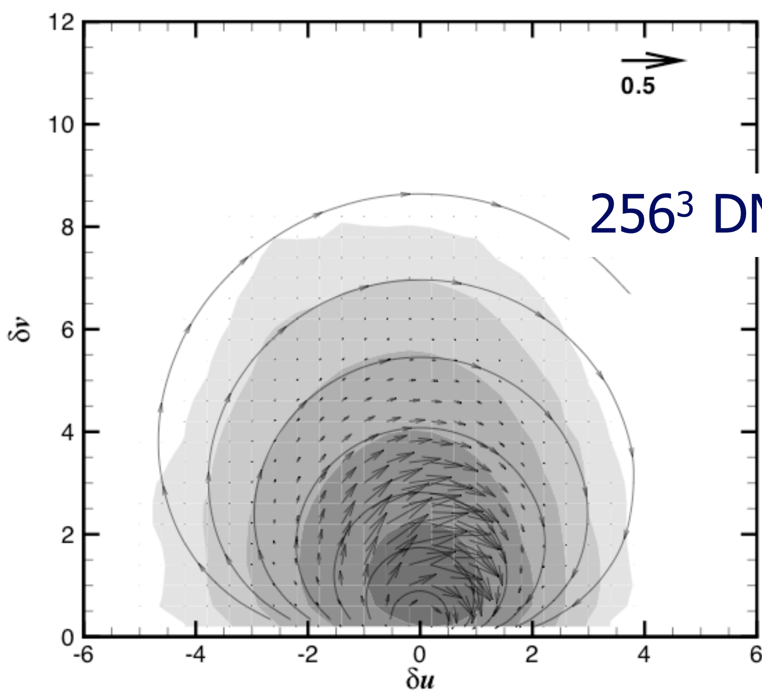
$$\begin{cases} \dfrac{d}{dt}\delta u = \dfrac{1}{\ell}\left(\delta v^2 - \delta u^2\right) - \dfrac{2}{3}Q\ell + H_{mn}\dfrac{r_m r_i}{r^2}\ell \\[2ex] \dfrac{d}{dt}\delta v = -\dfrac{1}{\ell}\delta u \delta v \quad + \quad H_{mn}\dfrac{r_m e_n}{r}\ell, \end{cases}$$

$$H_{mn} = \left(\frac{\partial^2}{\partial x_m \partial x_k}\left[p\delta_{kn} - \tau_{kn}^{SGS} + 2\nu \tilde{S}_{kn}\right]\right)^{anisotropic}$$

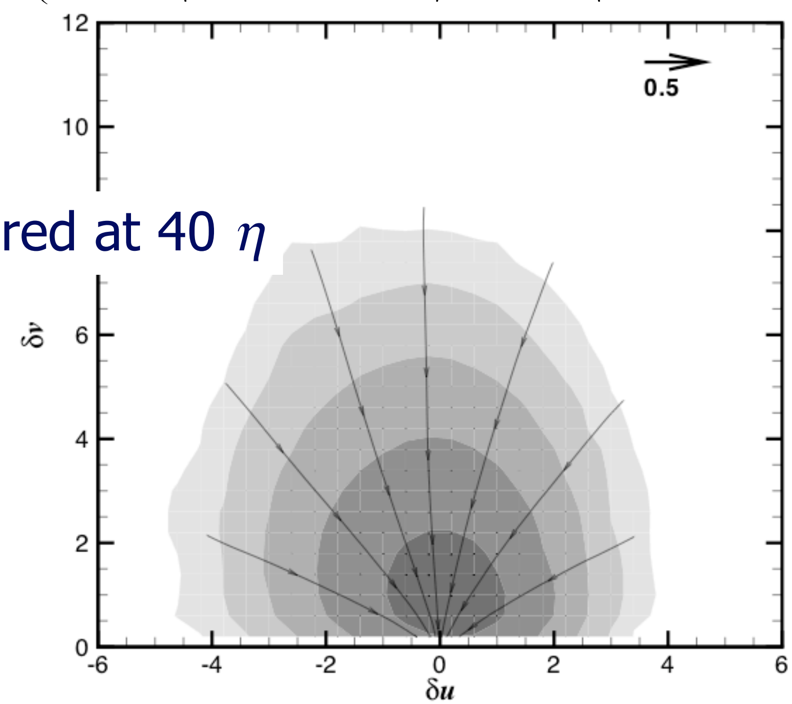
In equation for joint PDF of ($\delta u,\delta v$), terms enter as conditional averages:

$$\left(P(\delta u,\delta v)\left\langle \frac{1}{\ell}\left(\delta v^2 - \frac{1}{3}\delta u^2\right) \middle| \delta u,\delta v \right\rangle, P(\delta u,\delta v)\left\langle -\frac{2}{\ell}\delta u \delta v \middle| \delta u,\delta v \right\rangle \right)$$

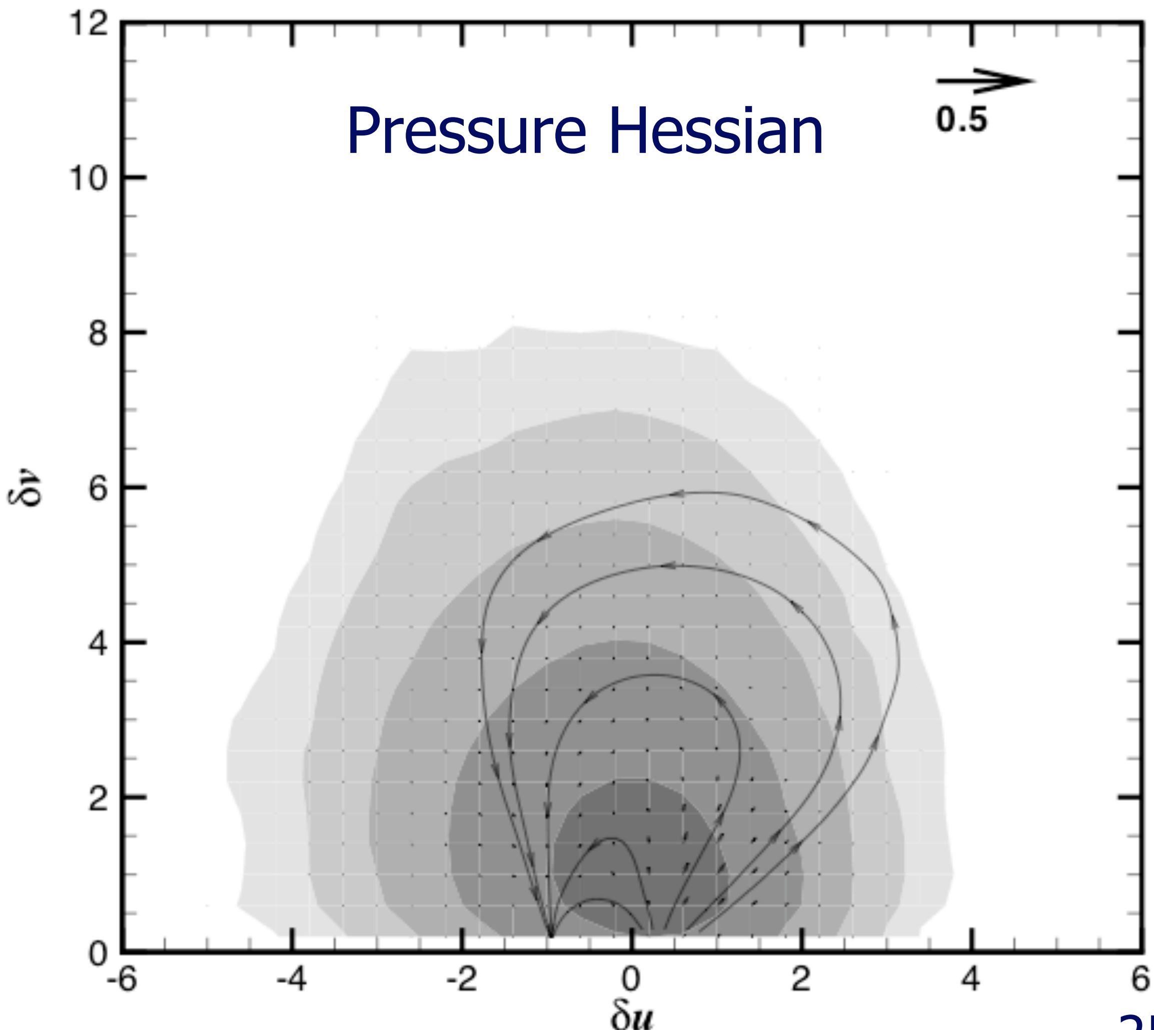
$$2\nu\left(P(\delta u,\delta v)\left\langle \frac{r_m r_i}{r^2}\ell \frac{\partial^2 \tilde{S}_{kn}}{\partial x_m \partial x_k} \middle| \delta u,\delta v \right\rangle, P(\delta u,\delta v)\left\langle \frac{r_m e_n}{r}\ell \frac{\partial^2 \tilde{S}_{kn}}{\partial x_m \partial x_k} \middle| \delta u,\delta v \right\rangle \right)$$



256^3 DNS, filtered at 40 η

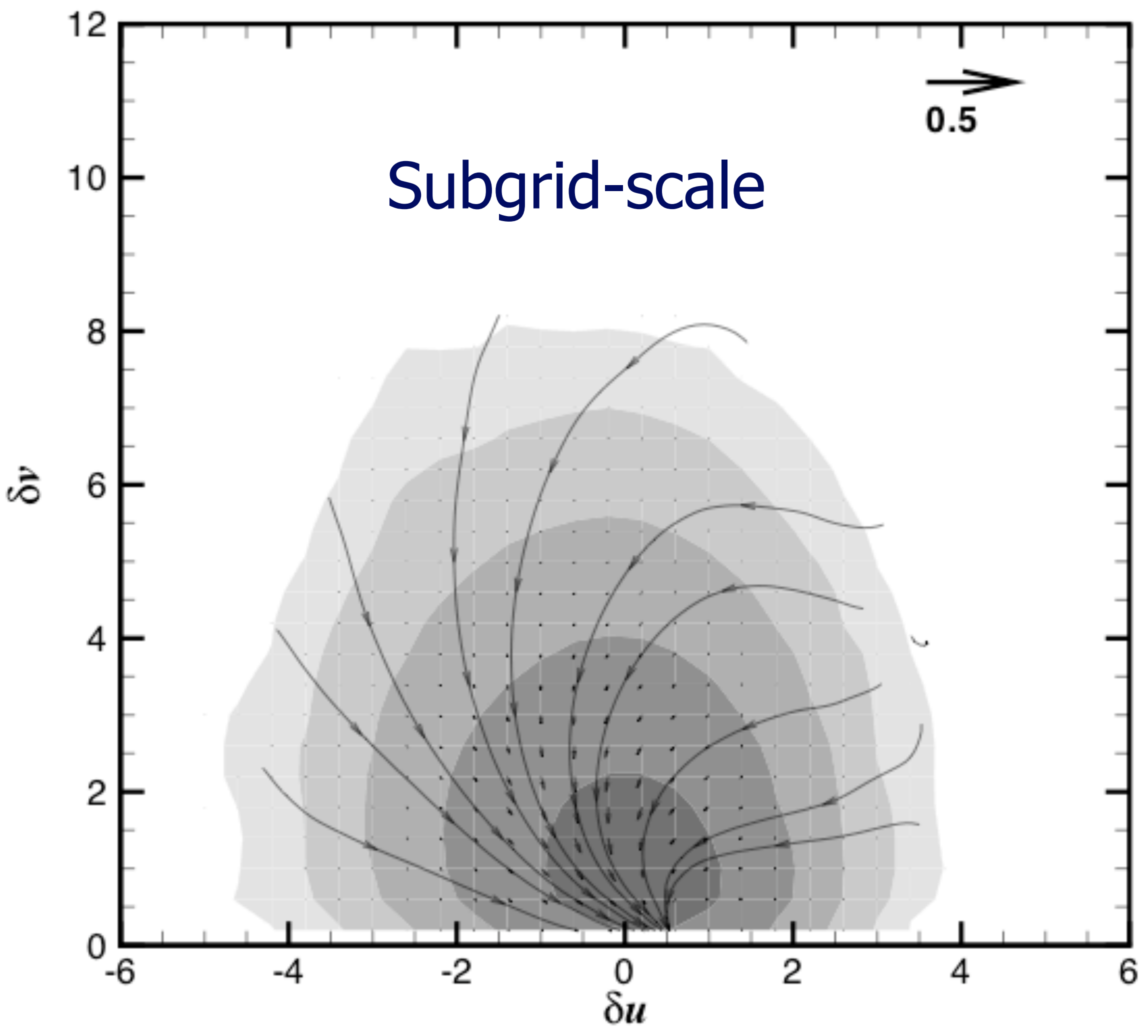


$$\left(P(\delta u,\delta v)\left\langle \frac{r_m r_n}{r^2}\ell\frac{\partial^2\tilde{p}}{\partial x_m\partial x_n}\middle|\delta u,\delta v\right\rangle, P(\delta u,\delta v)\left\langle \frac{r_m e_n}{r}\ell\frac{\partial^2\tilde{p}}{\partial x_m\partial x_n}\middle|\delta u,\delta v\right\rangle\right)$$

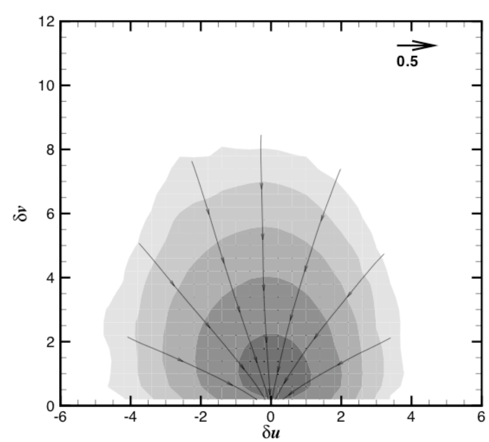


256^3 DNS, filtered at 40 η

$$-\left(P(\delta u,\delta v)\left\langle \frac{r_m r_n}{r^2}\,\ell\,\frac{\partial^2 \tau_{kn}^{SGS}}{\partial x_m \partial x_k}\middle|\delta u,\delta v\right\rangle, P(\delta u,\delta v)\left\langle \frac{r_m e_n}{r}\,\ell\,\frac{\partial^2 \tau_{kn}^{SGS}}{\partial x_m \partial x_k}\middle|\delta u,\delta v\right\rangle \right)$$



viscous



256^3 DNS, filtered at 40 η

Summary:

- We have found a higher-dimensional variant of the Burgers' 1-D gradient-steepening equation:

$$\frac{d}{dt}\delta u = -\delta u^2 \quad \longrightarrow \quad \begin{cases} \frac{d}{dt}\delta u = \delta v^2 - \delta u^2 \\ \frac{d}{dt}\delta v = -2\delta u \ \delta v \end{cases}$$

See: Yi & Meneveau, Phys. Rev. Lett. **95**, 164502, Oct. 2005

- Describes simple "mechanism" of self and cross amplification of velocity increments, leading to skewness in longitudinal and flare-up of long tails in transverse velocity increments.
- Due to Lagrangian nature, a measure correction must be applied to evolving PDFs.
- Q-δu^2 correlation: Predicts correct trends as function of dimensionality.
- Quantitative predictions - stationary PDFs as function of scale: need to take into account the effects of neglected terms. Analysis of DNS shows these effects are "non-trivial" (& each term different trends).